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A NEW NUMERICAL SCHEME APPLIED ON RE-VISITED NONLINEAR MODEL OF PREDATOR-PREY BASED ON DERIVATIVE WITH NON-LOCAL AND NON-SINGULAR KERNEL

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Abstract. A new concept of dynamical system of predator-prey model is presented in this paper. The model takes into account the memory of interaction between the prey and predator due to the inclusion of fractional differentiation. In addition, the model takes into account the inherent disposition of a prey or predator toward hunting or defending in time. Analysis of existence and uniqueness of the solutions is presented. A numerical method is used to generate some simulations as the fractional orders change from one to zero. A new traveling waves patterns are obtained.

1. **Introduction.** The main aim of modelling is to describe a physical problem using mathematical equations and give a better prediction. Mathematical models are employed in almost all branches of science natural sciences including chemistry, physics, biochemistry, biology, earth science and meteorology. Also these tools are also utilised in engineering disciplines including artificial intelligence and computer science. In addition, social sciences subjects such as political science, sociology, economics and psychology do use mathematical tools most extremely. Noting that the primary purpose of these exercises to help explain a system and to investigate the effects of different components, and have a more accurate predictions about the behaviour of the given physical problem under investigation. In general these models can takes several forms for instance statistical models, differential equations, game theory and dynamical system. In this work we are interesting in mathematical models describing the dynamical system of prey-predators models . We shall recall that within a given bionetwork, predation is a natural interface where a hunter feeds on it's pray. The model has attracted many scholars within the field of mathematical biology and other fields of applied mathematics. The first model was constructed

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in the theory of autocatalytic chemical reactions in 1910 by Alfred J. Lotka [11]. Then the well-known logistic equation was suggested by Pierre François Verhulst [12]. Many other researchers have modified these models to better obtained a real behaviour of the physical system [13], [14], [10], [4], [1], [8], [9] . However, their models do not take into account the space component, the nonlocality of the interaction between prey and predators, their model memory effect of the interaction, for example a new behaviour of a prey that survive the attack of the particular predator. The famous mathematical model constructed to describe this dynamical system is given as

$$\begin{aligned} \frac{du(t)}{dt} &= -\frac{\chi u(t)v(t)}{1+\chi hu(t)} + ru(t)\left(1-\frac{u(t)}{k}\right), \qquad (1) \\ \frac{dv(t)}{dt} &= \frac{\beta u(t)v(t)}{1+\chi hu(t)} - dv(t). \end{aligned}$$

Here $u(t)$ and $v(t)$ are the densities of predators and prey respectively.

Model (1) was recently modified to:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} + (1-\rho_1)v_1\partial_x u(x,t) &= -\frac{\chi u(x,t)v(x,t)}{1+\chi hu(x,t)} + ru(x,t)\left(1-\frac{u(x,t)}{k}\right), \quad (2) \\ \frac{\partial v(x,t)}{\partial t} + (1-\rho_2)v_2\partial_x u(x,t) &= \frac{\beta u(x,t)v(x,t)}{1+\chi hu(x,t)} - dv(x,t). \end{aligned}$$

Where v_1, v_2 average velocity of predator and prey respectively are, ρ_1, ρ_2 are suggested to be the factor of reduction of run due to diseases, age and starvation.

The above model equation (2) is more complex than the original version however does accurately replicate the physical problem, for the following reasons: The velocity of a prey or predator is function of time and space, the nonlocality is not taking into account here, the model does not takes into account the inherent disposition of a prey or predator toward hunting or defending in time. The memory of previous interaction is not considered, therefore a new model is needed.

2. **Motivation.**

In broad-spectrum, model involvedness encompasses an interchange between simplicity and accuracy of the model. An idea was suggested in modelling known as Occam's razor with the standard predominantly applicable to modelling, the main ideology being that among models with approximately equal prediction power, only the simplest one can be considered. Although added involvedness habitually enhances the levelheadedness of a model, nevertheless it can make the model problematic to understand and analyse, this can lead to computational problems, including numerical instability. Some well-known scientist like Thomas Kuhn who debates that as science advances, clarifications have a tendency to become multifaceted beforehand paradigm shift offers fundamental simplification. For instance when modelling the dynamic interaction between predator and prey, we could implant each natural behaviour of predator or prey into our model and would thus acquire an almost white-box model of the dynamical system. Nevertheless the mathematical analysis of such model would effectively inhibit the usage of such model. Adding to this, the uncertainty would increase the efficiency of the model and one could obtain a better prediction. It is important to note that, the advance of technology nowadays could help analysis more complicated models.

3. **Atangana-Baleanu derivative in Caputo sense.** In this section, we present the definitions of the new fractional derivative with no singular and nonlocal kernel [5], [6], [3], [2], [7] .

Definition 1. Let $f \in H^1(a,b)$, $b > a$, $\alpha \in [0,1]$ then, the definition of the new fractional derivative (Atangana-Baleanu derivative in Caputo sense)is given as:

$$ {}_{a}^{ABC}D_t^{\alpha}\left(f\left(t\right)\right) = \frac{B(\alpha)}{1-\alpha}\int\limits_a^t f^{'}(x)E_{\alpha}\left[-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right]dx. \tag{3}$$

Of course B has the same properties as in Caputo and Fabrizio case . The above definition will be helpful to real world problem and also will have great advantage when using Laplace transform to solve some physical problem with initial condition.

Definition 2. Let $f \in H^1(a,b)$, $b > a$, $\alpha \in [0,1]$ and not necessary differentiable then, the definition of the new fractional derivative (Atangana-Baleanu fractional derivative in Riemann-Liouville sense) is given as:

$$ {}_{a}^{ABR}D_t^{\alpha}\left(f\left(t\right)\right) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int\limits_a^t f(x)E_{\alpha}\left[-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right]dx. \tag{4}$$

Definition 3. The fractional integral associate to the new fractional derivative with non-local kernel is defined as:

$$ {}_{a}^{AB}I_t^{\alpha}\left\{f(t)\right\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int\limits_a^t f(y)(t-y)^{\alpha-1}dy. \tag{5}$$

When alpha is zero we recover the initial function and if also alpha is 1, we obtain the ordinary integral.

4. **New extended model with Atangana-Balenau derivative in Caputo sense.** Our new extended model is given by (6) with nonlocal and nonsingular derivative of ABC.

$$\begin{aligned} & {}_{0}^{ABC}D_t^{\alpha}u(x,t) + (1-\rho_1)v_1(x,t)\partial_x u(x,t) + \rho_1\phi_1\partial_t u(x,t) \\ = & -\frac{\chi u(x,t)v(x,t)}{1+\chi h u(x,t)} + r u(x,t)\left(1-\frac{u(x,t)}{k}\right), \\ & {}_{0}^{ABC}D_t^{\mu}v(x,t) + (1-\rho_2)v_2(x,t)\partial_x u(x,t) + \rho_2\phi_2\partial_t v(x,t) \\ = & \frac{\beta u(x,t)v(x,t)}{1+\chi h u(x,t)} - d v(x,t). \end{aligned} \tag{6}$$

Therefore the additional terms $\rho_1\phi_1\partial_t u(x,t)$ and $\rho_2\phi_2\partial_t v(x,t)$ contribute to the natural deposition of a given set of preys and predators as the prey can develop natural ability to defend themselves and also the predator can enhance the skill of hunting. The inclusion of fractional differentiation helps to account for the non-locality of the dynamical system. We also suggest that, the speeds of prey and predator depend on time and space.

5. **Existence and uniqueness of solutions.** In this section, we will research of existence and uniqueness of solutions. Also positive solution will be discussed. Let we consider $X = C[a, b]$ the Banach space of every continuous real functions defined in the closed set $[a, b]$, which contain the sub norm and Z be the shaft defined as: $Z = \{u, v \in X, u(x, t) \geq 0 \text{ and } v(x, t) \geq 0, a \leq t \leq b\}$. Now we present the following Banach fixed-point theorem that will be used for the existence of solutions .

Definition 4. Let E be a real Banach space with a cone H. H initiates a restricted order $\leq$ in E in the succeeding approach [15]

$$x \leq y \Rightarrow y - x \in H.$$

For every x, $y \in E$ the order interval is defined as $\langle a, b\rangle = \{f \in E : a \leq f \leq b\}$. A cone K is denoted normal, if one can find a positive constant j such that $h, d \in K$, $\Phi < h < d \Rightarrow \|h\| \leq j \|d\|$, where Φ denotes the zero element of K.

Theorem 1 [15]**.** *Let H be a closed set subspace of a Banach space of D. Let G be a contraction mapping with Lipschitz constant $g < 1$ from H to H. Thus G possesses a fixed-point t^* in H. In addition, if t_0 is a random point in H and $\{t_n\}$ is a sequence defined by $t_{n+1} = Gt_n$ $(n = 0, 1, 2 \ldots)$, then for a large number n, t_n tends to t^* in H and $d(t_n, t^*) \leq \frac{g^n}{(1-g)} d(t_1, t_0)$.*

In the rest of section we will consider new model with ABC derivative as below:

$$\begin{aligned} {}_0^{ABC}D_t^{\alpha} u(x,t) &= -\frac{\chi u(x,t)v(x,t)}{1+\chi h u(x,t)} + r u(x,t)\left(1 - \frac{u(x,t)}{k}\right) \\ &\quad -(1-\rho_1)v_1(x,t)\partial_x u(x,t) - \rho_1\phi_1\partial_t u(x,t), \\ {}_0^{ABC}D_t^{\mu} v(x,t) &= \frac{\beta u(x,t)v(x,t)}{1+\chi h u(x,t)} - d v(x,t) \\ &\quad -(1-\rho_2)v_2(x,t)\partial_x u(x,t) - \rho_2\phi_2\partial_t v(x,t). \end{aligned} \tag{7}$$

Now applying the AB fractional integral on equation (7), we obtain the followings;

$$\begin{aligned} u(x,t) - u(x,0) &= \frac{1-\alpha}{AB(\alpha)}\left\{ \begin{array}{c} -\frac{\chi u(x,t)v(x,t)}{1+\chi h u(x,t)} + r u(x,t)\left(1 - \frac{u(x,t)}{k}\right) \\ -(1-\rho_1)v_1(x,t)\partial_x u(x,t) - \rho_1\phi_1\partial_t u(x,t) \end{array}\right\} \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} \\ &\quad .\left\{ \begin{array}{c} -\frac{\chi u(x,\tau)v(x,\tau)}{1+\chi h u(x,\tau)} + r u(x,\tau)\left(1 - \frac{u(x,\tau)}{k}\right) \\ -(1-\rho_1)v_1(x,\tau)\partial_x u(x,\tau) - \rho_1\phi_1\partial_t u(x,\tau) \end{array}\right\} d\tau, \end{aligned} \tag{8}$$

$$\begin{aligned} v(x,t) - v(x,0) &= \frac{1-\mu}{AB(\mu)}\left\{ \begin{array}{c} \frac{\beta u(x,t)v(x,t)}{1+\chi h u(x,t)} - d v(x,t) \\ -(1-\rho_2)v_2(x,t)\partial_x u(x,t) - \rho_2\phi_2\partial_t v(x,t) \end{array}\right\} \\ &\quad + \frac{\mu}{B(\mu)\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1} \\ &\quad .\left\{ \begin{array}{c} \frac{\beta u(x,\tau)v(x,\tau)}{1+\chi h u(x,\tau)} - d v(x,\tau) \\ -(1-\rho_2)v_2(x,\tau)\partial_x u(x,\tau) - \rho_2\phi_2\partial_t v(x,\tau) \end{array}\right\} d\tau. \end{aligned} \tag{9}$$

Then we will use (8)-(9) to show existence of equation (7). Necessary lemmas for existence of solutions given with lemma 1 and lemma 2 below.

Lemma 1. *The mapping $G : H \to H$ defined as*

$$\begin{aligned} Gu(x,t) &= \frac{1-\alpha}{AB(\alpha)} s(x,t,u(x,t)) \\ &\quad + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} s(x,\tau,u(x,\tau)) d\tau \end{aligned} \tag{10}$$

and

$$\begin{aligned} Gv(x,t) &= \frac{1-\mu}{AB(\mu)} s(x,t,v(x,t)) \\ &\quad + \frac{\mu}{AB(\mu)\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} s(x,\tau,v(x,\tau)) d\tau. \end{aligned}$$

For simplicity we use

$$s(x,t,u(x,t)) = \left\{ \begin{array}{c} -\frac{\chi u(x,t) v(x,t)}{1+\chi h u(x,t)} + r u(x,t) \left(1 - \frac{u(x,t)}{k}\right) \\ -(1-\rho_1) v_1(x,t) \partial_x u(x,t) - \rho_1 \phi_1 \partial_t u(x,t) \end{array} \right\}$$

and

$$s(x,t,v(x,t)) = \left\{ \begin{array}{c} \frac{\beta u(x,t) v(x,t)}{1+\chi h u(x,t)} - d v(x,t) \\ -(1-\rho_2) v_2(x,t) \partial_x u(x,t) - \rho_2 \phi_2 \partial_t v(x,t) \end{array} \right\}.$$

Lemma 2. *Let $M \subset H$ be bounded implying, we can find $p, r > 0$ for system such that,*

$$\begin{aligned} \|u(x,t_2) - u(x,t_1)\| &\leq p \|t_2 - t_1\|, \\ \|v(x,t_2) - v(x,t_1)\| &\leq r \|t_2 - t_1\|, \ \forall u, v \in M. \end{aligned} \tag{11}$$

Then $\overline{G(M)}$ is compact.

Proof. Let $P = \max\left\{\frac{1-\alpha}{AB(\alpha)} + s(x,t,u(x,t))\right\}$, $0 \leq u(x,t) \leq K$. For $u(x,t) \in M$ then we have the following.

$$\begin{aligned} \|Gu(x,t)\| &= \frac{1-\alpha}{AB(\alpha)} \|s(x,t,u(x,t))\| \\ &\quad + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|s(x,\tau,u(x,\tau))\| d\tau \\ &\leq \frac{1-\alpha}{AB(\alpha)}.P + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}.P\frac{t^\alpha}{\alpha} \\ &\leq \frac{1-\alpha}{AB(\alpha)}.P + \frac{\alpha t^\alpha P}{AB(\alpha)\Gamma(\alpha+1)}. \end{aligned} \tag{12}$$

And similary let we consider second equation, If $R = \max\left\{\frac{1-\mu}{AB(\mu)} + s(x,t,v(x,t))\right\}$, $0 \leq v(x,t) \leq L$. For $v(x,t) \in M$ then we have the following.

$$\|Gv(x,t)\| \leq \frac{1-\mu}{AB(\mu)} \|s(x,t,v(x,t))\| \tag{13}$$

$$
\begin{aligned}
&+\frac{\mu}{AB(\mu)\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}\left\|s(x,\tau,v(x,\tau))\right\|d\tau \\
\leq\; & \frac{1-\mu}{AB(\mu)}.R+\frac{\mu R t^{\mu}}{AB(\mu)\Gamma(\mu+1)}.
\end{aligned}
$$

So (12)-(13) implies the function G is bounded. On the rest of section, we will consider $u(x,t)\in M$, t_1,t_2 and $t_1<t_2$, then for a given $\epsilon>0$, if $|t_2-t_1|<\delta$. Then,

$$
\begin{aligned}
\left\|Gu(x,t_2)-Gu(x,t_1)\right\| \leq\; & \frac{1-\alpha}{AB(\alpha)}\left\|s(x,t_2,u(x,t_2))-s(x,t_1,u(x,t_1))\right\| \qquad (14)\\
&+\left\|\begin{array}{c} \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^{t_2}(t_2-\tau)^{\alpha-1}\left\|s(x,\tau,u(x,\tau))\right\|d\tau \\ -\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^{t_1}(t_1-\tau)^{\alpha-1}\left\|s(x,\tau,u(x,\tau))\right\|d\tau \end{array}\right\| \\
\leq\; & \frac{1-\alpha}{AB(\alpha)}\left\|s(x,t_2,u(x,t_2))-s(x,t_1,u(x,t_1))\right\| \\
&+\frac{\alpha K}{AB(\alpha)\Gamma(\alpha)}\left\{\int_0^{t_2}(t_2-y)^{\alpha-1}dy-\int_0^{t_1}(t_1-y)^{\alpha-1}dy\right\}.
\end{aligned}
$$

We first start with the integral part.

$$
\begin{aligned}
&\int_0^{t_2}(t_2-y)^{\alpha-1}dy-\int_0^{t_1}(t_1-y)^{\alpha-1}dy \qquad (15)\\
=\; & \int_0^{t_1}\left\{(t_1-y)^{\alpha-1}-(t_2-y)^{\alpha-1}\right\}dy+\int_{t_1}^{t_2}(t_2-y)^{\alpha-1}dy \\
=\; & 2\frac{(t_2-t_1)^{\alpha}}{\alpha}
\end{aligned}
$$

Now we investigate the following;

$$
\begin{aligned}
&\left\|s(x,t_2,u(x,t_2))-s(x,t_1,u(x,t_1))\right\| \qquad (16)\\
=\; & \left\|\begin{array}{c} u(x,t_2)\left[-\frac{\chi v(x,t_2)}{1+\chi h u(x,t_2)}\right]-u(x,t_1)\left[-\frac{\chi v(x,t_1)}{1+\chi h u(x,t_1)}\right] \\ +\frac{r}{k}\left(u(x,t_2)-u(x,t_1)\right)\left[1-u(x,t_2)-u(x,t_1)\right] \\ +(\rho_1-1)\left[v_1(x,t_2)\partial_x u(x,t_2)-v_1(x,t_1)\partial_x u(x,t_1)\right] \\ -\rho_1\phi_1\left[\partial_t u(x,t_2)-\partial_t u(x,t_1)\right] \end{array}\right\|.
\end{aligned}
$$

Since solutions are bounded, we can find different positive constants, A,B,C,D, such that for all t. Also, with $\left[-\frac{\chi v(x,t_2)}{1+\chi h u(x,t_2)}\right]<1$ and $\left[-\frac{\chi v(x,t_1)}{1+\chi h u(x,t_1)}\right]$ are satisfied and using Lipchitz condition of the derivative , the above (16) can be rewritten as below.

$$
\begin{aligned}
&\left\|s(x,t_2,u(x,t_2))-s(x,t_1,u(x,t_1))\right\| \qquad (17)\\
\leq\; & \left\|u(x,t_2)-u(x,t_1)\right\|A+\frac{r}{k}B\left\|u(x,t_2)-u(x,t_1)\right\| \\
&+(\rho_1-1)C\left\|u(x,t_2)-u(x,t_1)\right\|-\rho_1\phi_1 D\left\|u(x,t_2)-u(x,t_1)\right\|
\end{aligned}
$$

$$\begin{aligned}
&\leq \underbrace{\left\{A + \frac{r}{k}B + (\rho_1 - 1)C - \rho_1\phi_1 D\right\}}_{y} \|u(x,t_2) - u(x,t_1)\| \\
&\leq y.p\,\|t_2 - t_1\| \\
&\leq F\,\|t_2 - t_1\|.
\end{aligned}$$

Now putting (17)-(15) in (14) we obtain:

$$\begin{aligned}
\|Gu(x,t_2) - Gu(x,t_1)\| &\leq \frac{1-\alpha}{AB(\alpha)}F\,\|t_2 - t_1\| + \frac{\alpha K}{AB(\alpha)\Gamma(\alpha)}2\frac{\|t_2 - t_1\|^{\alpha}}{\alpha} \quad (18)\\
&\leq \frac{1-\alpha}{AB(\alpha)}F\,\|t_2 - t_1\| + \frac{2\alpha K}{AB(\alpha)\Gamma(\alpha+1)}\|t_2 - t_1\|^{\alpha},
\end{aligned}$$

$$\delta = \frac{\epsilon}{\frac{1-\alpha}{AB(\alpha)}F + \frac{2\alpha K}{AB(\alpha)\Gamma(\alpha+1)}}. \quad (19)$$

Such that $\|Gu(x,t_2) - Gu(x,t_1)\| \leq \epsilon$ is satisfied.

If we apply similar steps to second equation we can obatin as below:

$$\|Gv(x,t_2) - Gv(x,t_1)\| \leq \frac{1-\mu}{AB(\mu)}Q\,\|t_2 - t_1\| + \frac{2\mu L}{AB(\mu)\Gamma(\mu+1)}\|t_2 - t_1\|^{\alpha}. \quad (20)$$

For each $\epsilon > 0$, we can find

$$\delta = \frac{\epsilon}{\frac{1-\mu}{AB(\mu)}Q + \frac{2\mu L}{AB(\mu)\Gamma(\mu+1)}}. \quad (21)$$

So $\|Gv(x,t_2) - Gv(x,t_1)\| \leq \epsilon$. Henceforth $G(M)$ is equi-continuous and from the meaning of Arzela-Ascoli theorem, $\overline{G(M)}$ is compact. □

Theorem 2. *$S : [a,b] \times [0,\infty) \to [0,\infty)$ be a continuous function and $S(x,t,.)$ increasing for each t in $[a,b]$. Let us assume that one can find m,n satisfying $K(D)m \leq S(x,t,m)$, $K(D)n \geq S(x,t,n)$, $0 \leq m(x,t) \leq n(x,t)$, $a \leq t \leq b$. Then our new equation has a positive solution.*

Proof. The fixed-point of the operator G is needed to be considered. With framework of lemma 1, the considered operator $G : H \to H$ is completely continuous. Let us choose two arbitrary densities of population of predator in the u_1 and u_2 in H satisfying $u_1 \leq u_2$ and also densities of population of prey in the v_1 and v_2 in H satisfying $v_1 \leq v_2$ then, by assuming that S is a positive function, then followings are satified

$$\begin{aligned}
Gu_1(x,t) &\leq \frac{1-\alpha}{AB(\alpha)}\|s(x,t,u_1(x,t))\| \quad (22)\\
&\quad + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\|s(x,\tau,u_1(x,\tau))\|\,d\tau \\
&\leq Gu_2(x,t)
\end{aligned}$$

and

$$\begin{aligned}
\|Gv_1(x,t)\| &\leq \frac{1-\mu}{AB(\mu)}\|s(x,t,v_1(x,t))\| \quad (23)\\
&\quad + \frac{\mu}{AB(\mu)\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}\|s(x,\tau,v_1(x,\tau))\|\,d\tau \\
&\leq Gv_2(x,t).
\end{aligned}$$

Henceforth the mapping G is increasing. By the conjecture, we get $Gn \geq n$, $Gm \leq m$. So the operator $G : \langle m, n\rangle \to \langle m, n\rangle$ is compact within the framework of lemma 2 and continuous in view of lemma 1. Since H is a normal cone of G. □

5.1. **Uniqueness of solution.** In this section, we will investigate the uniqueness of solutions. To succes this, let we consider equations like below:

$$\begin{aligned}
&\left\| Gu_1(x,t) - Gu_2(x,t) \right\| \qquad (24)\\
\leq\ & \left\| \begin{array}{c} \frac{1-\alpha}{AB(\alpha)}\left(s(x,t,u_1(x,t)) - s(x,t,u_2(x,t))\right) \\ \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int\limits_0^t (t-\tau)^{\alpha-1}\left(s(x,\tau,u_1(x,\tau)) - s(x,\tau,u_2(x,\tau))\right)d\tau \end{array} \right\| \\
\leq\ & \frac{1-\alpha}{AB(\alpha)}\left\| s(x,t,u_1(x,t)) - s(x,t,u_2(x,t)) \right\| \\
& + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int\limits_0^t (t-\tau)^{\alpha-1}\left\| s(x,\tau,u_1(x,\tau)) - s(x,\tau,u_2(x,\tau)) \right\| d\tau \\
\leq\ & \frac{1-\alpha}{AB(\alpha)} y \left\| u_1(x,t) - u_2(x,t) \right\| \\
& + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)} y \int\limits_0^t (t-\tau)^{\alpha-1}\left\| u_1(x,\tau) - u_2(x,\tau) \right\| d\tau,
\end{aligned}$$

$$\begin{aligned}
&\left\| Gu_1(x,t) - Gu_2(x,t) \right\| \qquad (25)\\
\leq\ & \left\{ \frac{1-\alpha}{AB(\alpha)} y + \frac{\alpha y b^{\alpha}}{AB(\alpha)\Gamma(\alpha+1)} \right\} \left\| u_1(x,t) - u_2(x,t) \right\|.
\end{aligned}$$

And similary,

$$\begin{aligned}
&\left\| Gv_1(x,t) - Gv_2(x,t) \right\| \qquad (26)\\
\leq\ & \left\{ \frac{1-\mu}{AB(\mu)} z + \frac{\mu z b^{\mu}}{AB(\mu)\Gamma(\mu+1)} \right\} \left\| v_1(x,t) - v_2(x,t) \right\|.
\end{aligned}$$

Therefore if the following conditions holds

$$\left\{ \frac{1-\alpha}{AB(\alpha)} y + \frac{\alpha y b^{\alpha}}{AB(\alpha)\Gamma(\alpha+1)} \right\} < 1 \qquad (27)$$

and

$$\left\{ \frac{1-\mu}{AB(\mu)} z + \frac{\mu z b^{\mu}}{AB(\mu)\Gamma(\mu+1)} \right\} < 1, \qquad (28)$$

then mapping G is a contraction, which implies fixed point, thus the new model has a unique positive solution.

6. **New numerical approximation of fractional integral.** In the done by Atangana and Dumitru , it was revealed that partial differential equations based on the ABC derivative can be equivalently converted to fractional integral equations. However the AB fractional integral is based on the Riemann-Liouville fractional derivative, this implies researchers may have choice while dealing with fractional partial differential equations with new trends of fractional calculus. In this section, the discritization of AB integral is presented.

In this work, we consider $t \in [0,T]$, $\alpha \in (0,1)$ and $x \in [a,b]$. To do the discritization of the AB-integral we start by discritizing the time domains $[0,T]$ by replacing grid over the whole domain.

$\tau = \frac{T}{h}$, t_k, $k = 0,1,2,3,...,n$, where $t_k = k\tau$, $k = 0,1,2,3,...,n$. The same with space domain $[a,b]$, $\xi = \frac{b}{m}$, x_j, $j = 0,1,2,3,...,m$, where $x_j = j\xi$ or $k = 0,1,2,3,...,n$ and $j = 0,1,2,3,...,m$.

$$\begin{aligned} I_t^\alpha f(x_j,t_k) &= \frac{1-\alpha}{2AB(\alpha)}(f_j^{k+1}+f_j^k) \\ &\quad + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}(t_k-y)^{\alpha-1}f(x_j,y)dy \\ &= \frac{1-\alpha}{2AB(\alpha)}(f_j^{k+1}+f_j^k) \\ &\quad + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}(t_k-y)^{\alpha-1}\frac{f_j^{i+1}+f_j^i}{2}dy \\ &\quad + R_{k,\alpha}^i, \\ I_t^\alpha f(x_j,t_k) &= \frac{1-\alpha}{2AB(\alpha)}(\frac{f_j^{k+1}+f_j^k}{2}) \\ &\quad + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{k-1}\left(\frac{f_j^{i+1}+f_j^i}{2}\right)\left[\begin{array}{c}(k-i)^\alpha \\ -(k-i+1)^\alpha\end{array}\right]dy + R_{k,\alpha}^i. \end{aligned} \tag{29}$$

Here

$$\begin{aligned} R_{k,\alpha}^i &= \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}(t_k-y)^{\alpha-1}\frac{f(x_j,y)-f(x_j,t_{i+1})}{(t_k-y)^{1-\alpha}}dy \\ &= \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}(t_k-y)^{\alpha-1}\frac{f'(x_j,\varphi_i)-f(y-t_{i+1})}{(t_k-y)^{1-\alpha}}dy, \\ y &< \varphi_i < t_{i+1}, \end{aligned} \tag{30}$$

where One can see that the $R_{k,\alpha}^i$ is bounded if we consider

$$\begin{aligned} \left|R_{k,\alpha}^i\right| &\le \frac{t^\alpha}{AB(\alpha)\Gamma(\alpha)}\max_{0\le t\le t_k}\left|f'(x_i,t)\right|\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}(t_k-y)^{\alpha-1}dy \\ &\le \frac{\sum}{AB(\alpha)\Gamma(\alpha)}t_k^\alpha\max_{t\in(0,t_{i+1})}\left|f'(x_i,t)\right|. \end{aligned} \tag{31}$$

Lemma 3. *Let* $\beta_i^\alpha = (i+1)^\alpha - i^\alpha$, $i = 0,1,2,...,n$, *we have if* $f(x,t) \in C'([0,T]\times[a,b])$, *then*

$${}_0^{AB}I_t^\alpha f(x_i,t_k) = \frac{\tau^\alpha}{2AB(\alpha)\Gamma(\alpha)}\sum_{i=0}^{k-1}b_i^\alpha\left(f_j^{k-i-1}+f_j^{k-i}\right), \tag{32}$$

where $\left|R_{k,\alpha}^i\right| \le \tau B t_k^\alpha$, $k = 0,1,2,...,n$ [16] .

Here we evaluate the derivative of AB respect to time. By definition

$$D_t\, {}_0^{AB}I_t^{\alpha} f(x,t) = \frac{1-\alpha}{AB(\alpha)}\partial_t f(x,t) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\frac{d}{dt}\int_0^t \frac{f(x,y)}{(t-y)^{1-\alpha}}dy. \tag{33}$$

We chose here

$$\partial_t f(x,t_k) = f(x,t_{k+1}) - f(x,t_k),$$
$$z^{'}(t) = \frac{f(x,\Delta t+t)-f(x,t)}{\Delta t} = \frac{f(x,t+\tau)-f(x,t)}{\tau}. \tag{34}$$

For $k = 0,1,2,3,...,n$, the following is obtain

$$\begin{aligned} D_t\, {}_0^{AB}I_t^{\alpha} f(x,t_k) &= \frac{1-\alpha}{AB(\alpha)}\left[f(x,t_{k+1})-f(x,t_k)\right] \qquad (35)\\ &\quad + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\left[\int_0^{t_{k+1}} \frac{f(x,y)}{(t_{k+1}-y)^{1-\alpha}}dy - \int_0^{t_k} \frac{f(x,y)}{(t_k-y)^{1-\alpha}}dy\right]. \end{aligned}$$

Nevertheless

$$\begin{aligned} &\int_0^{t_{k+1}} \frac{f(x,y)}{(t_{k+1}-y)^{1-\alpha}}dy - \int_0^{t_k} \frac{f(x,y)}{(t_k-y)^{1-\alpha}}dy \qquad (36)\\ &= \int_0^{\tau} \frac{f(x,y)}{(t_k-y)^{1-\alpha}} + I^{\alpha}z(t_k), \end{aligned}$$

where also

$$\int_0^{\tau} \frac{f(x,y)}{(t_k-y)^{1-\alpha}}dy = \int_0^{\tau} \frac{f(x,\tau)}{(t_k-y)^{1-\alpha}}dy + \int_0^{\tau} \frac{f(x,y)-f(x,\tau)}{(t_k-y)^{1-\alpha}}dy, \tag{37}$$

thus

$$D_t\, {}_0^{AB}I_t^{\alpha} f(x,t_k) = \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^{\tau} \frac{f(x,y)}{(t_k-y)^{1-\alpha}}dy + {}_0^{AB}I^{\alpha}z(t_k). \tag{38}$$

If we consider the function $f(x,t) \in C^2[0,T]$ then

$$\begin{aligned} \left|\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^{\tau} \frac{f(x,y)-f(x,\tau)}{(t_k-y)^{1-\alpha}}dy\right| &= \left|\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_0^{\tau} \frac{\partial f(x,\xi)(y-\tau)}{(t_k-y)^{1-\alpha}}dy\right| \\ &\leq \frac{\tau^{\alpha+1}}{AB(\alpha)\Gamma(\alpha)}\beta_{k-1}^{\alpha}\max_{t\in(0,\tau]}\left|\partial_\xi f(x,\xi)\right|, \end{aligned}$$

where $0 < \xi < \tau$. This leads us to have

$${}_0^{AB}I_t^{\alpha} z(t_k) = \frac{1-\alpha}{AB(\alpha)}z(t_k) + \frac{\tau^{\alpha}}{AB(\alpha)\Gamma(\alpha)}\sum_{i=1}^{k-1} b_{k-1}^{\alpha} z(t_{k+i}) + R_{k,\alpha}^{1}\ , \tag{39}$$

with

$$\left|R_{k,\alpha}^{1}\right| \leq \frac{\tau}{AB(\alpha)\Gamma(\alpha)}t_{k-1}^{\alpha}\max_{t\in[0,t_{k-1}]}\left|z^{'}\right| \tag{40}$$

$$\begin{aligned} &\leq \frac{\tau}{AB(\alpha)\Gamma(\alpha)} t_{k-1}^{\alpha} \max_{t\in[0,t_{k-1}]} \left|f'(x,t+\tau)-f'(x,t)\right| \\ &= \frac{\tau}{AB(\alpha)\Gamma(\alpha)} t_{k-1}^{\alpha} \max_{0\leq t\leq t_k} \left\| f''_{\xi_k}(x,\xi_k)\right\|, \ \ 0<\xi_k<t_k\ .. \end{aligned}$$

Finally the following numerical approximation is obtained.

$$\begin{aligned} & D_t\ {}_0^{AB}I_t^{\alpha} f(x,t_k) \\ = & \frac{1-\alpha}{AB(\alpha)}\left(\frac{f(x,t_{k+1})+f(x,t_k)}{2}\right) \\ & +\frac{\tau^{\alpha}}{AB(\alpha)\Gamma(\alpha)}\left[\begin{array}{c} f(x,t_k)+f(x,t_{k-1}) \\ +\sum_{i=1}^{k-1}\left(b_i^{\alpha}-b_{i-1}^{\alpha}\right)\left[f(x,t_{k-j})-f(x,t_{k-j-1})\right] \end{array}\right] \\ & +R_{k,\alpha}^3, \end{aligned} \tag{41}$$

with $\left|R_{k,\alpha}^3\right| \leq Bb_{k-1}^{\alpha}\tau^{\alpha+1}$.

6.1. **Application to the new model of predator prey.** In this section, we develop the numerical solution of the new model of predator-prey. Let

$$\begin{aligned} f_1(x,t,v,u) &= -\frac{\chi u(x,t)v(x,t)}{1+\chi h u(x,t)} + r u(x,t)\left(1-\frac{u(x,t)}{k}\right) \\ &\quad -(1-\rho_1)v_1(x,t)\partial_x u(x,t) - \rho_1\phi_1\partial_t u(x,t), \end{aligned} \tag{42}$$

$$\begin{aligned} f_2(x,t,v,u) &= \frac{\beta u(x,t)v(x,t)}{1+\chi h u(x,t)} - dv(x,t) \\ &\quad -(1-\rho_2)v_2(x,t)\partial_x u(x,t) - \rho_2\phi_2\partial_t v(x,t). \end{aligned}$$

Then the equation can be converted to

$$\begin{aligned} u(x,t)-u(x,0) &= {}_0^{AB}I_t^{\alpha} f_1(x,t,v,u), \\ v(x,t)-v(x,0) &= {}_0^{AB}I_t^{\alpha} f_2(x,t,v,u). \end{aligned} \tag{43}$$

Using the numerical scheme suggested here, we have

$$\begin{aligned} & \frac{u_i^{k+1}+u_i^k}{2} - u(x_i,0) \\ = & \frac{1-\alpha}{2AB(\alpha)}\left[f_1(x_{i+1},t_{k+1},v_i^{k+1},u_i^{k+1}) + f_1(x_i,t_k,v_i^k,u_i^k)\right] \\ & +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\sum_{j=0}^{k-1}\left[f_1(x_{i+1},t_{j+1},v_i^{j+1},u_i^{j+1}) + f_1(x_i,t_j,v_i^j,u_i^j)\right] \\ & .\left[(k-j)^{\alpha}-(k-j-i)^{\alpha}\right], \end{aligned} \tag{44}$$

$$\begin{aligned} & \frac{v_i^{k+1}+v_i^k}{2} - v(x_i,0) \\ = & \frac{1-\alpha}{2AB(\alpha)}\left[f_2(x_{i+1},t_{k+1},v_i^{k+1},u_i^{k+1}) + f_2(x_i,t_k,v_i^k,u_i^k)\right] \\ & +\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\sum_{j=0}^{k-1}\left[f_2(x_{i+1},t_{j+1},v_i^{j+1},u_i^{j+1}) + f_2(x_i,t_j,v_i^j,u_i^j)\right] \\ & .\left[(k-j)^{\alpha}-(k-j-i)^{\alpha}\right]. \end{aligned} \tag{45}$$

6.2. **Graphical simulations.** Using the numerical scheme of (44)-(45) for the new model of predator-prey above, we obtain the following numerical simulations. We give these simulations in figure 1 for $\alpha = 0.05$, in figure 2 for $\alpha = 0.5$, in figure 3 for $\alpha = 0.8$ and finally in figure 4 for $\alpha = 1$.

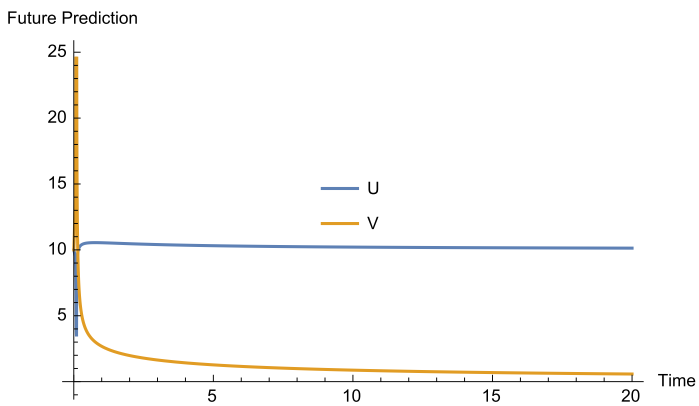

FIGURE 1. Numerical solution for $\alpha = 0.05$.

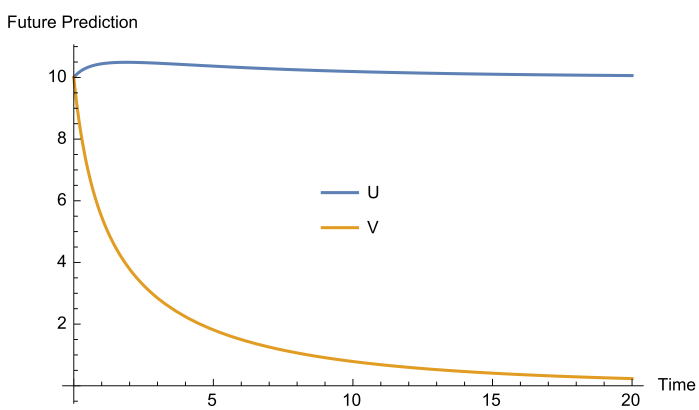

FIGURE 2. Numerical solution for $\alpha = 0.5$.

7. **Conclusion.** Recently, based on the generalized Mittag-Leffler function, a new concept of fractional differentiation was suggested and applied in many fields of science. The aim of the new concept of to well describe natural occurrences as existing derivatives based on power law have some limitation while modelling real world problems. In this paper a new model of predator-prey is suggested, which include the velocity and the animal instinct. The time-fractional based on the ABC derivative is used. The existence and uniqueness of couple solutions are investigated. A new numerical scheme based on Atangana-Baleanu fractional integral is suggested. The new numerical scheme is used to solve numerically the new model.

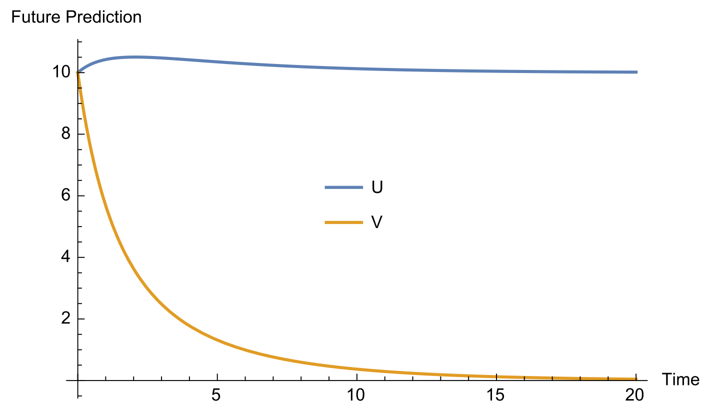

FIGURE 3. Numerical solution for $\alpha = 0.8$.

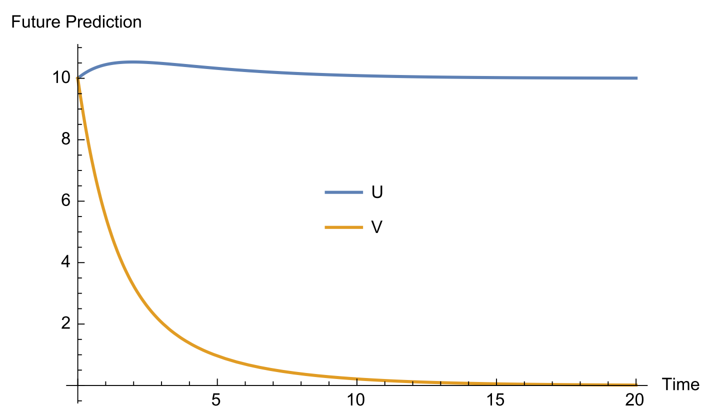

FIGURE 4. Numerical solution for $\alpha = 1$.

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REFERENCES

[1] P. A. Abrams and L. R. Ginzburg, The nature of predation: Prey dependent, ratio dependent or neither?, *Trends in Ecology & Evolution*, **15** (2000), 337–341.
[2] B. S. T. Alkahtani, Chua's circuit model with Atangana–Baleanu derivative with fractional order, *Chaos, Solitons and Fractals,* **89** (2016), 547–551.
[3] O. J. J. Alkahtani, Comparing the Atangana–Baleanu and Caputo–Fabrizio derivative with fractional order: Allen Cahn model, *Chaos, Solitons and Fractals*, **89** (2016), 552–559.
[4] R. Arditi and L. R. Ginzburg, Coupling in predator-prey dynamics: Ratio dependence, *Journal of Theoretical Biology*, **139** (1989), 311–326.
[5] A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model, *Therm. Sci.*, **20** (2016), 763–769.
[6] A. Atangana and I. Koca, Chaos in a simple nonlinear system with Atangana–Baleanu derivatives with fractionalorder, *Chaos Solitons Fractals*, **89** (2016), 447–454.
[7] A. Atangana and D. Baleanu, Caputo-Fabrizio applied to groundwater flow within a confined aquifer, *J Eng Mech,* **143** (2016), D4016005.

[8] A. Atangana, Non validity of index law in fractional calculus: A fractional differential operator with Markovian and non-Markovian properties, *Physica A: Statistical Mechanics and its Applications*, **505** (2018), 688–706.
[9] A. Atangana and J. F. Gomez Aguila, Decolonisation of fractional calculus rules: Breaking commutativity and associativity to capture more natural phenomena, *The European Physical Journal Plus*, **133** (2018), 166.
[10] G. Gandolfo, Giuseppe Palomba and the Lotka–Volterra equations, *Rendiconti Lincei*, **19** (2008), 347–257.
[11] A. J. Lotka, Contribution to the theory of periodic reaction, *J. Phys. Chem.*, **14** (1910), 271–274.
[12] P. H. Verhulst, Notice sur la loi que la population poursuit dans son accroissement, *Corresp. Mathématique et Physique,* **10** (1838), 113–121.
[13] V. Volterra, *Variations and Fluctuations of the Number of Individuals in Animal Species Living Together in Animal Ecology*, Chapman, R.N. (ed), McGraw–Hill, 1931.
[14] V. Volterra, Variazioni e fluttuazioni del numero d'individui in specie animali conviventi, *Mem. Acad. Lincei Roma*, **2** (1926), 31–113.
[15] T. Yamamoto and X. Chen, An existence and nonexistence theorem for solutions of nonlinear systems and its application to algebraic equations, *Journal of Computational and Applied Mathematics*, **30** (1990), 87–97.
[16] P. Zhuang, F. Liu, I. Turner and ·V. Anh, Galerkin finite element method and error analysis for the fractional cable equation, *Appl. Math. Comput.*, **217** (2010), 2534–2545.

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