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ORIGINAL ARTICLE

Investigation of the spectrum of singular Sturm–Liouville operators on unbounded time scales

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Abstract

In this article, we consider the self-adjoint singular operators associated with the Sturm–Liouville expression

$$Ly := -\left[p(t)\, y^{\Delta}(t)\right]^{\nabla} + q(t)\, y(t), \ t \in (-\infty, \infty)_{\mathbb{T}}.$$

on time scale $\mathbb{T}$. Some conditions are given for this operator to have a discrete spectrum. Further, we investigate the continuous spectrum of this operator. We also prove that the regular Sturm–Liouville operator on time scale is semi-bounded from below which is not studied in literature yet.

Keywords Sturm–Liouville operator · Unbounded time scales · Splitting method · Discrete spectrum · Continuous spectrum

Mathematics Subject Classification 34N05 · 47A10 · 47B25

1 Introduction

Nowadays, dynamic equations on time scales has attracted much interest because it unites the theory of differential and difference equations. In the context, it has led to several important applications, e.g., in the study of heat transfer, insect population models, epidemic models stock market,and neural networks (see [1–4]).

On the other hand, in the literature, there is a few study concerning spectral theory of the Sturm–Liouville operators on time scales. In [5], the authors studied a second-

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order Sturm–Liouville operator with a spectral parameter in the boundary condition on bounded time scales. They proved the completeness of the system of eigenvectors and associated vectors of the dissipative Sturm–Liouville operators on bounded time scales. In [6], Guseinov established some expansion results for a Sturm–Liouville problems on time scale. In [7], the author constructed a space of boundary values for minimal symmetric singular second-order dynamic operators on semi-infinite and infinite time scales in limit-point and limit-circle cases. He gave a description of all maximal dissipative, maximal accumulative, selfadjoint, and other extensions of such symmetric operators in terms of boundary conditions. In [8], the author studied the maximal dissipative second-order dynamic operators on semi-infinite time scale. In [9], the author studied an operator defined by the second order Sturm–Liouville equation on an unbounded time scale. For such an operator he gave characterisations of the domains of its Krein-von Neumann and Friedrichs extensions by using the recessive solution. In [10], the author proved the completeness of the system of eigenfunctions for dissipative Sturm–Liouville operators. In [11], Agarwal et al. gave an oscillation theorem and establish Rayleigh's principle for Sturm–Liouville eigenvalue problems on time scales with separated boundary conditions. In [12], Huseynov investigated the classical concepts of Weyl limit point and limit circle cases for second order linear dynamic equations on time scales. In [13], the authors obtained a min–max characterization of the eigenvalues of the Sturm–Liouville problems on time scales, and various eigenfunction expansions for functions in suitable function spaces. In [14], the author examined Green's function for an nth -order focal boundary value problem on time scales. In [15], the authors studied properties of the spectrum of a Sturm–Liouville operator on semi-infinite time scales.

In the operator theory, one of the important operator class is the class of self-adjoint differential operators. This operators play an important role in quantum mechanics. The spectrum of such operators depend on the behavior of the coefficients of the corresponding differential expression. This problem has been investigated by many mathematicians (see [15–25]).

The purpose of this paper is to extend some results obtained in [15] to the case of singular Sturm–Liouville dynamic equation

$$Ly := -\left[p(t)\, y^{\Delta}(t)\right]^{\nabla} + q(t)\, y(t) = \lambda y(t)\,,\ t \in (-\infty, \infty)_{\mathbb{T}}, \tag{1}$$

where p, q are real-valued continuous functions on unbounded time scale $\mathbb{T}$ and $p(t) \neq 0$ for all $t \in \mathbb{T}$. We prove that the regular Sturm–Liouville operator on time scale is semi-bounded from below. Using the splitting method [17], we will give some conditions for the self-adjoint operator associated with the singular expression (1) to have a discrete spectrum. We also investigate the continuous spectrum of this operator.

2 Preliminaries

Now, we recall some necessary fundamental concepts of time scales, and we refer to [1,6,9,12,26–32] for more details.

Definition 1 Let $\mathbb{T}$ be a time scale. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \ t \in \mathbb{T}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \ t \in \mathbb{T}.$$

If $\sigma(t) > t$, we say that t is right scattered, while if $\rho(t) < t$, we say that t is left scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. We introduce the sets $\mathbb{T}^k, \ \mathbb{T}_k, \ \mathbb{T}^*$ which are derived form the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Finally, $\mathbb{T}^* = \mathbb{T}^k \cap \mathbb{T}_k$.

Definition 2 A function f on $\mathbb{T}$ is said to be Δ-differentiable at some point $t \in \mathbb{T}^k$ if there is a number $f^{\Delta}(t)$ such that for every $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \quad s \in U.$$

Analogously one may define the notion of ∇-differentiability of some function using the backward jump ρ. One can show (see [26])

$$f^{\Delta}(t) = f^{\nabla}(\sigma(t)), \, f^{\nabla}(t) = f^{\Delta}(\rho(t))$$

for continuously differentiable functions.

Example 3 If $\mathbb{T} = \mathbb{R}$, then we have

$$\sigma(t) = t, \ f^{\Delta}(t) = f'(t).$$

If $\mathbb{T} = \mathbb{Z}$, then we have

$$\sigma(t) = t + 1, \ f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t).$$

If $\mathbb{T} = q^{\mathbb{N}_0} = \left\{q^k : q > 1, \ k \in \mathbb{N}_0\right\}$, ($\mathbb{N}_0 := \{0, 1, 2, \ldots\}$) then we have

$$\sigma(t) = qt, \ f^{\Delta}(t) = \frac{f(qt) - f(t)}{qt - t}.$$

Definition 4 Let $f : \mathbb{T} \to \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \to \mathbb{R}$, such that $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$, then F is a Δ-antiderivative of f. In this case the integral is given by the formula

$$\int_a^b f(t)\,\Delta t = F(b) - F(a) \ \text{ for } a, b \in \mathbb{T}.$$

Analogously one may define the notion of ∇-antiderivative of some function.

Let $L^2_\nabla(\mathbb{T})$ be the space of all functions defined on $\mathbb{T}$ such that

$$\|f\| := \left(\int_a^b |f(t)|^2 \nabla t\right)^{1/2} < \infty.$$

Let $\mathbb{T}$ be a time scale such that $\inf \mathbb{T} = -\infty$ and $\sup \mathbb{T} = \infty$. We will denote $\mathbb{T}$ also as $(-\infty, \infty)_{\mathbb{T}}$.

The space $L^2_\nabla(-\infty, \infty)_{\mathbb{T}}$ is a Hilbert space with the inner product (see [33])

$$(f, g) := \int_{-\infty}^{\infty} f(t)\overline{g(t)}\nabla t, \quad f, g \in L^2_\nabla(-\infty, \infty)_{\mathbb{T}}.$$

The *Wronskian* of $y(.)$, $z(.)$ is defined by (see [7,9,26])

$$W_t(y, z) := p(t)\left[y(t) z^\Delta(t) - y^\Delta(t) z(t)\right], \quad t \in \mathbb{T}. \tag{2}$$

Definition 5 Let D_A denote a subset of the complex Hilbert space H. A linear operator A is said to be Hermitian if, for all $x, y \in D_A$, $(Ax, y) = (x, Ay)$ holds. A Hermitian operator with a domain D_A of definition dense in H is called a symmetric operator. An operator A^* defined on $D_{A^*} \subseteq H$ is called the adjoint of symmetric operator A if for all $x \in D_A$, $y \in D_{A^*}$, $(Ax, y) = (x, A^* y)$. An operator with a domain D_A of definition dense in H is said to be self-adjoint if $A = A^*$. An operator A is said to be compact if it maps every bounded set into a compact set (see [34]).

Definition 6 A complex number λ is called a regular point of the linear operator A acting in complex Hilbert space H if

(R1) *The inverse* $R_\lambda(A) = (A - \lambda I)^{-1}$ (*where* I *is the identity operator in* H) *exists, and*
(R2) $R_\lambda(A)$ *is bounded operator defined on the whole space* H.

Let

(R3) $R_\lambda(A)$ *is defined on a set which dense* H.

The operator $R_\lambda(A)$ is then called the *resolvent* of the operator A. All non-regular points λ are called points of the *spectrum* of the operator A.

The point spectrum or discrete spectrum $\sigma_p(A)$ is the set such that $R_\lambda(A)$ does not exist. A $\lambda \in \sigma_p(A)$ is called an eigenvalue of A. The spectrum of the operator A is said to be *purely discrete* if it consists of a denumerable set of eigenvalues with no finite point of accumulation.

The *continuous spectrum* $\sigma_c(A)$ is the set such that $R_\lambda(A)$ exists and satisfies **(R3)** but not **(R2)**.

The *residual spectrum* $\sigma_r(A)$ is the set such that $R_\lambda(A)$ exists but does not satisfy **(R3)** (see [35]).

Theorem 7 [34] *All self-adjoint extensions of a closed, symmetric operator which has equal and finite deficiency indices have one and same continuous spectrum.*

Theorem 8 [35] *The residual spectrum $\sigma_r(A)$ of a self-adjoint linear operator acting on a complex Hilbert space H is empty.*

Definition 9 [34] A symmetric operator A is said to be semi-bounded from below if there is a number m such that, for all $x \in D_A$, the inequality

$$(Ax, x) \geq m \|x\|^2$$

holds. Similarly, if for all $x \in D_A$, there is a number M such that the inequality

$$(Ax, x) \leq M \|x\|^2$$

holds, then A is said to be semi-bounded from above.

Theorem 10 [34] *If a symmetric operator A with finite deficiency indices (n, n) satisfies the condition*

$$(Ax, x) \geq m \|x\|^2, \ x \in D_A,$$

or the condition

$$(Ax, x) \leq M \|x\|^2, \ x \in D_A,$$

then the part of the spectrum of every self-adjoint extension of A which lies to the left of m or to the right of M can consist of only a finite number of eigenvalues and the sum of their multiplicities does not exceed n.

Definition 11 [34] The direct sum $A_1 \oplus A_2$ of two operators A_1, A_2 in the spaces H_1, H_2 is an operator in the space $H_1 \oplus H_2$ of all ordered pairs $\{x_1, x_2\}$, $x_1 \in H_1$, $x_2 \in H_2$; its domain of definition is the set of all ordered pairs $\{x_1, x_2\}$, $x_1 \in D_{A_1}$, $x_2 \in D_{A_2}$, and

$$(A_1 \oplus A_2)\{x_1, x_2\} = \{A_1 x_1, A_2 x_2\}.$$

It is easily seen that if A_1 and A_2 are each self-adjoint operators, then their direct sum $A_1 \oplus A_2$ is also a self-adjoint operator.

3 Main results

Let us consider the linear set $D_{\max}$ consisting of all vectors $y \in L^2_\nabla(-\infty, \infty)_\mathbb{T}$ such that y and py^∇ are locally Δ absolutely continuous functions on $(-\infty, \infty)_\mathbb{T}$ and $Ly \in L^2_\nabla(-\infty, \infty)_\mathbb{T}$. We define the *maximal operator* $L_{\max}$ on $D_{\max}$ by the equality $L_{\max} y = Ly$.

For every $y, z \in D_{\max}$ we have Green's formula (or Lagrange's identity)

$$\int_a^b (Ly)(t)\overline{z(t)}\nabla t - \int_a^b y(t)\overline{(Lz)(t)}\nabla t$$
$$= [y, z](b) - [y, z](a), \; a, b \in (-\infty, \infty)_{\mathbb{T}}, \; a < b,$$

where $[y, z](t)$ denotes the Lagrange bracket defined by

$$[y, z](t) := p(t)(y(t)\overline{z^{\nabla}(t)} - y^{\nabla}(t)\overline{z(t)})$$

(see [7,9,31]).

It is clear that from Green's formula limits

$$[y, z](\infty) := \lim_{t\to\infty} [y, z](t), \;\; [y, z](-\infty) := \lim_{t\to\infty} [y, z](t)$$

exist and are finite for all $y, z \in D_{\max}$.

Let $D_{\min}$ be the linear set of all vectors $y \in D_{\max}$ satisfying the conditions

$$[y, z](-\infty) = [y, z](\infty) = 0, \tag{3}$$

for arbitrary $z \in D_{\max}$. The operator $L_{\min}$, that is the restriction of the operator $L_{\max}$ to $D_{\min}$ is called the *minimal operator* and the equalities $L_{\max} = L^*_{\min}$ holds. Further (it follows from (3)) $T_{\min}$ is closed symmetric operator with deficiency indices $(1, 1)$ or $(2, 2)$ [7,9,34,36].

Let us consider the linear set $D^a_{\max}$ consisting of all vectors $y \in L^2_{\nabla}(-a, a)_{\mathbb{T}}$ ($a \in \mathbb{T}$, $a > 0$) such that y and py^{∇} are Δ absolutely continuous functions on $[-a, a]_{\mathbb{T}}$ and $Ly \in L^2_{\nabla}(-a, a)_{\mathbb{T}}$. We define the *maximal operator* $L^a_{\max}$ on $D^a_{\max}$ by the equality $L^a_{\max}y = Ly$. Let D_a be the linear set of all vectors $y \in D^a_{\max}$ satisfying the conditions

$$y(-a) = y(a) = 0. \tag{4}$$

We define the *operator* $\mathcal{L}_a$ on D_a by the equality $\mathcal{L}_a y = L^a_{\max} y$.

Theorem 12 *If $p(t) > 0$ ($t \in [-a, a]_{\mathbb{T}}$), $a > 0$), then the regular operator $\mathcal{L}_a$ acting in $L^2_{\nabla}(-a, a)_{\mathbb{T}}$ is semi-bounded from below. Further, the negative part of the spectrum of $\mathcal{L}_a$ consists of not more that a finite number of negative eigenvalues of finite multiplicity.*

Proof By integration by parts, we get

$$(\mathcal{L}_a y, y) = \int_{-a}^a Ly\overline{y}\nabla t = \int_{-a}^a \left[-\left[py^{\Delta}\right]^{\nabla} + q(t)y\right]\overline{y}\nabla t$$
$$= \int_{-a}^a \left[-\left[py^{\Delta}\right]^{\nabla}\overline{y} + q(t)\left|y\right|^2\right]\nabla t$$
$$= \int_{-a}^a \left[\left|py^{\Delta}\right|^2 + q(t)\left|y\right|^2\right]\nabla t.$$

We set

$$v(t,\xi)=\begin{cases}1, & \xi\leq t\\ 0, & \xi>t\end{cases},$$

and

$$H(\xi,\eta)=-\int_{-a}^{a}q(t)\,v(t,\xi)\,v(t,\eta)\,\nabla t.$$

For $y\in D_a$, we have

$$y(t)=\int_{-a}^{a}\frac{v(t,\xi)\left(py^{\Delta}\right)(\xi)}{p(\xi)}\nabla\xi.$$

Hence, we get

$$\begin{aligned}(\mathcal{L}_a y,y)=&\int_{-a}^{a}\frac{\left|(py^{\Delta})(\xi)\right|^2}{p(\xi)}\nabla\xi\\ &-\int_{-a}^{a}\int_{-a}^{a}\frac{H(\xi,\eta)\,(py^{\Delta})(\xi)\left(p\overline{y}^{\Delta})(\eta)\right)}{p(\xi)\,p(\eta)}\nabla\xi\nabla\eta.\end{aligned}\tag{5}$$

Let $L^2_{\nabla,p}(-a,a)_{\mathbb{T}}$ be the Hilbert space of all complex-valued functions defined on $[-a,a]_{\mathbb{T}}$ with the inner product

$$(f_1,f_2)_1=\int_{-a}^{a}f_1(t)\,\overline{f_2(t)}\frac{1}{p(t)}\nabla t.$$

In $L^2_{\nabla,p}(-a,a)_{\mathbb{T}}$ we consider the integral operator K with the symmetric kernel $H(\xi,\eta)$:

$$Kf=\int_{-a}^{a}\frac{H(\xi,\eta)}{p(\eta)}f(\eta)\,\nabla\eta,$$

where

$$\int_{-a}^{a}\int_{-a}^{a}\frac{|H(\xi,\eta)|^2}{p(\xi)\,p(\eta)}\nabla\xi\nabla\eta<\infty,$$

i.e., $H(\xi,\eta)$ is a Hilbert-Schmidt kernel. Since the symmetric kernel $H(\xi,\eta)$ is a Hilbert-Schmidt kernel, the integral operator K is a compact operator in the space $L^2_{\nabla,p}(-a,a)_{\mathbb{T}}$. Thus it has a purely discrete spectrum. Let $\varphi_1,\varphi_2,\varphi_3,\ldots$ be a complete orthonormal system of eigenfunctions of the operator K and $\lambda_1,\lambda_2,\lambda_3,\ldots$ be the corresponding eigenvalues. From the Hilbert-Schmidt theorem, we get

$$(Kf, f)_1 = \sum_{k=1}^{\infty} \lambda_k \left|(f, \varphi_k)_1\right|^2.$$

As $k \to \infty$, we have $\lambda_k \to 0$. Then there is a certain number N such that $\lambda_k < 1$ for $k > N$. For $(f, \varphi_k)_1 = 0,\ k = 1, 2, \ldots, N$, we have

$$(Kf, f)_1 = \sum_{k=N+1}^{\infty} \lambda_k \left|(f, \varphi_k)_1\right|^2 \leq \sum_{k=N+1}^{\infty} \left|(f, \varphi_k)_1\right|^2,$$

that is,

$$(Kf, f)_1 \leq (f, f)_1. \tag{6}$$

Let $\mathcal{D}$ denote the manifold of all functions $y \in D_a$ which satisfy the conditions

$$\left(pf^{\Delta}, \varphi_k\right)_1 = 0,\ k = 1, 2, \ldots, N,\ y \in D_a.$$

By (6), we have, for $y \in \mathcal{D}$,

$$\int_{-a}^{a}\int_{-a}^{a} \frac{H(\xi, \eta)\left(py^{\Delta}\right)(\xi)\left(p\overline{y}^{\Delta}(\eta)\right)}{p(\xi)\,p(\eta)} \nabla\xi\nabla\eta$$
$$\leq \left(Kpy^{\Delta}, py^{\Delta}\right)_1 \leq \left(py^{\Delta}, py^{\Delta}\right)_1 = \int_{-a}^{a} \frac{\left|\left(py^{\Delta}\right)(\xi)\right|^2}{p(\xi)} \nabla\xi.$$

From the equality (5), we obtain

$$(\mathcal{L}_a y, y) \geq 0.$$

On the other hand, the dimension of the manifold D_a modulo $\mathcal{D}$ is N, and consequently, the operator $\mathcal{L}_a$ is semi bounded from below on the whole manifold D_a. It is clear that the operator $\mathcal{L}_a$ is a self-adjoint operator. By Theorem 10, we get the desired result. □

Let H' denotes the set of all functions f from $L^2_{\nabla}(-\infty, \infty)_{\mathbb{T}}$ which vanish outside a finite interval $[\alpha, \beta] \subset (-\infty, \infty)_{\mathbb{T}}$ and $D'_{\min} = H' \cap D_{\min}$.

Further, let $L'_{\min}$ denote the restriction of the operator $L_{\min}$ to D'. Then $L_{\min}$ is the closure of the operator $L'_{\min}$, i.e., $\widetilde{L'_{\min}} = L_{\min}$ [34].

Now we restrict $D'_{\min}$ by imposing the additional conditions

$$y(-c) = y(c) = 0,$$

where c is fixed point of the interval $(0, \infty)_{\mathbb{T}}$. By this restriction, we obtain the manifold $D''_{\min}$.

The restriction $L''_{\min}$ of the operator $L'_{\min}$ to $D''_{\min}$ is called the splitting of the operator $L'_{\min}$ at the points $-c$ and c of the interval $(-\infty,\infty)_{\mathbb{T}}$. It is clear that

$$L''_{\min} = L'_1 \oplus \mathcal{L}_c \oplus L'_2, \tag{7}$$

i.e., the operator $L''_{\min}$ is the direct sum of three operators L'_1, $\mathcal{L}_c$ and L'_2 in the spaces $L^2_\nabla(-\infty,-c)_{\mathbb{T}}$, $L^2_\nabla(-c,c)_{\mathbb{T}}$ and $L^2_\nabla(c,\infty)_{\mathbb{T}}$, where L'_1, $\mathcal{L}_c$ and L'_2 are generated in these spaces from the Sturm–Liouville expression L in the same way as $L'_{\min}$ was.

If $L_1 = \widetilde{L'_1}$, and $L_2 = \widetilde{L'_2}$ are the closures of the operators L'_1 and L'_2, then (7) implies that

$$\widetilde{L''_{\min}} = L_1 \oplus \mathcal{L}_c \oplus L_2.$$

If we extend the symmetric operators L_1 and L_2 into self-adjoint operators $L_{1,s}$ and $L_{2,s}$ in the spaces $L^2_\nabla(-\infty,-c)_{\mathbb{T}}$, and $L^2_\nabla(c,\infty)_{\mathbb{T}}$ respectively, then the direct sum

$$A = L_{1,s} \oplus \mathcal{L}_c \oplus L_{2,s}$$

will be a self-adjoint extension of the symmetric operator $\widetilde{L''_{\min}}$. The spectrum of the operator A is the set-theoretic sum of the spectra of $L_{1,s}$, $\mathcal{L}_c$ and $L_{2,s}$.

Since the deficiency indices of the operator $\widetilde{L''_{\min}}$ are finite, by Theorem 7, all its self-adjoint extensions have one and the same continuous spectrum. Both the operator A and also each self-adjoint extension L_s of the operator $L_{\min}$ are such extensions. Hence, the continuous parts of spectrum of the two operators A and L_s coincide.

Therefore, we have the following theorem:

Theorem 13 *The continuous parts of the spectrum of every self-adjoint extension of the operator $L_{\min}$ is the set-theoretic sum of the continuous parts of the spectra of $L_{1,s}$, $\mathcal{L}_c$ and $L_{2,s}$, where $L_{1,s}$, $\mathcal{L}_c$ and $L_{2,s}$ have been obtained by the splitting of the operator $L_{\min}$.*

Theorem 14 *If*

$$\lim_{t\to\pm\infty} q(t) = +\infty \tag{8}$$

and

$$p(t) > 0,\ t \in (-\infty,\infty)_{\mathbb{T}} \tag{9}$$

then every self-adjoint extension L_s of the singular operator $L_{\min}$ has a purely discrete spectrum.

Proof Let $N > 0$ be an arbitrary number. From (8), one can choose numbers $-c$ and c such that

$$|q(t)| > N \text{ for } t \in (-\infty,\infty)_{\mathbb{T}} \setminus (-c,c). \tag{10}$$

By the condition (9), via integration by parts, we obtain ($y \in D_{L_1'}$)

$$
\begin{aligned}
\left(L_1' y, y\right) &= \int_{-\infty}^{-c} L y \overline{y} \nabla t = \int_{-\infty}^{-c} \left[-\left[p y^{\Delta}\right]^{\nabla} + q(t) y\right] \overline{y} \nabla t \\
&= \int_{-\infty}^{-c} \left[-\left[p y^{\Delta}\right]^{\nabla} \overline{y} + q(t) \left|y\right|^2\right] \nabla t \\
&= \int_{-\infty}^{-c} \left[p \left|y^{\Delta}\right|^2 + q(t) \left|y\right|^2\right] \nabla t > N \int_{-\infty}^{-c} \left|y\right|^2 \nabla t = N\left(y, y\right).
\end{aligned}
$$

Hence the operator L_1' is bounded from below and its closure L_1 is also bounded from below by the number N. Therefore, by Theorem 10, the half-axis $-\infty < \lambda < N$, contains no point of the continuous spectrum of the self-adjoint extension $L_{1,s}$ of L_1.

Similarly, by the condition (9), via integration by parts, we obtain ($y \in D_{L_2'}$)

$$
\begin{aligned}
\left(L_2' y, y\right) &= \int_{c}^{\infty} L y \overline{y} \nabla t = \int_{c}^{\infty} \left[-\left[p y^{\Delta}\right]^{\nabla} + q(t) y\right] \overline{y} \nabla t \\
&= \int_{c}^{\infty} \left[-\left[p y^{\Delta}\right]^{\nabla} \overline{y} + q(t) \left|y\right|^2\right] \nabla t \\
&= \int_{c}^{\infty} \left[p \left|y^{\Delta}\right|^2 + q(t) \left|y\right|^2\right] \nabla t > N \int_{c}^{\infty} \left|y\right|^2 \nabla t = N\left(y, y\right).
\end{aligned}
$$

Hence the operator L_2' is bounded from below and its closure L_2 is also bounded from below by the number N. Therefore, by Theorem 10, the half-axis $-\infty < \lambda < N$, contains no point of the continuous spectrum of the self-adjoint extension $L_{2,s}$ of L_2.

On the other hand, since the operator L_2 is regular and self-adjoint, the spectrum of $\mathcal{L}_c$ is purely discrete. Hence the half-axis $-\infty < \lambda < N$, contains no point of the continuous spectrum of $A = L_{1,s} \oplus \mathcal{L}_c \oplus L_{2,s}$.

By Theorem 7, every self-adjoint extension L_s of the operator $L_{\min}$ has this property. Since the number N is arbitrary, the spectrum of the operator L_s has no continuous part at all. □

Theorem 15 *Let*

$$\lim_{t \to \pm\infty} q\left(t\right) = M$$

and $p\left(t\right) > 0$ $(t \in (-\infty, \infty)_{\mathbb{T}})$. *Then the interval* $(-\infty, M)$ *contains no point of the continuous spectrum of any, self-adjoint extension* L_s *of the singular operator* $L_{\min}$*; on the contrary, any* L_s *can only have at most point-eigenvalues on this interval and these can have a point of accumulation only at* $\lambda = M$.

Proof If we decompose the operator at points $-c$ and c such that

$$q\left(t\right) > M - \varepsilon \text{ for } x \in (-\infty, \infty)_{\mathbb{T}} \setminus (-c, c),$$

then we obtain

$$\left(L_1' y, y\right) > (M - \varepsilon)(y, y).$$

Hence, the part of the spectrum of L_1 lying in the interval $(-\infty, M - \varepsilon)$ can consist only of a finite number of eigenvalues of finite multiplicity. Likewise, we obtain

$$\left(L_2' y, y\right) > (M - \varepsilon)(y, y).$$

Consequently, the part of the spectrum of L_2 lying in the interval $(-\infty, M - \varepsilon)$ can consist only of a finite number of eigenvalues of finite multiplicity. On the other hand, by Theorem 12, the operator L_2 is regular and bounded below. Hence its spectrum is purely discrete; and any point of accumulation of the spectrum L_2 can only be at $\lambda = +\infty$. Thus, from Theorem 13, we get the desired result. □

Now, we need following lemma.

Lemma 16 *If the interval $[\lambda_0 - \delta, \lambda_0 + \delta]$ contains no point of the spectrum of a self-adjoint operator A except perhaps for a finite number of eigenvalues each of finite multiplicity, and if Q is a bounded Hermitian operator satisfying the condition*

$$\|Q\| < \delta,$$

then the point λ_0 does not lie in the continuous part of the spectrum of the operator $A + Q$.

Proof See [34]. □

Theorem 17 *Let $p(t) \equiv 1$ and*

$$\lim_{t \to \pm\infty} |q(t)| = M$$

Then any interval of length greater than $2M$, of the positive half-axis contains continuous spectrum of any self-adjoint extension L_s of the singular operator $L_{\min}$.

Proof Suppose, contrary to our claim, that an interval $[\lambda_0 - \delta, \lambda_0 + \delta]$ of the half-axis $\lambda > 0$ contains no point of the continuous spectrum of L_s, $\delta > M$. Then, the operator may be decomposed, this interval would contain no point of the continuous spectrum of any self-adjoint extension of $L_{\min}$. If we choose the points $-c$ and c such that

$$|q(t)| \le M + \varepsilon < \delta \text{ for } |t| > c,$$

then, by Lemma 16, λ_0 can not belong to the continuous spectrum of the self-adjoint extension of the minimal operator generated by the expression $-\left[y^{\Delta}\right]^{\nabla}$ and the same boundary conditions. But this is contradiction because the continuous spectrum of last operator covers the whole of the positive half-axis. □

In particular, for $M = 0$ we have the following corollary.

Corollary 18 *Let $p(t) \equiv 1$ and*

$$\overline{\lim_{t\to\pm\infty}} |q(t)| = 0.$$

Then the whole positive half-axis is covered by the continuous spectrum of any self-adjoint extension L_s of the singular operator $L_{\min}$.

Corollary 19 *Let $p(t) \equiv 1$ and*

$$\overline{\lim_{t\to\pm\infty}} |q(t)| = \rho < \infty, \quad \underline{\lim_{t\to\pm\infty}} |q(t)| = \sigma > -\infty.$$

Then any interval, of length greater than $(\rho - \sigma)$, of the half-axis

$$\lambda > \frac{1}{2}(\rho + \sigma)$$

contains of the continuous spectrum of any self-adjoint extension L_s of the singular operator $L_{\min}$.

Proof For, if $q_1(t) = q(t) - \frac{1}{2}(\rho + \sigma)$, then

$$\overline{\lim_{t\to\pm\infty}} |q_1(t)| = \frac{1}{2}(\rho - \sigma),$$

and the result follows by replacing $q(t)$ by $q_1(t)$, i.e., by applying Theorem 17 to the operator $L_s - \frac{1}{2}(\rho + \sigma) I$. □

Example 20 Let $\mathbb{T} = \mathbb{R}$. The Hermite differential equation is given by

$$-y'' + t^2 y = \lambda y, \text{ for all } t \in (-\infty, \infty).$$

Since $p(t) \equiv 1$ and

$$\lim_{t\to\pm\infty} t^2 = \infty,$$

we can apply Thoerem 14. Thus the self-adjoint singular operator L_s corresponding to the equation $-y''+t^2y = \lambda y$ has a purely discrete spectrum. In fact, for all $n \in \mathbb{N}_0$ and for the eigenvalues $\lambda = 2n + 1$, this equation has the Hermite functions $e^{\left(-\frac{1}{2}t^2\right)} H_n(t)$ for solutions (eigenfunctions); please see [37, Chapter IV, Section 2].

Example 21 Consider the dynamic equation

$$-y^{\Delta\nabla} + e^{-t^2} y = \lambda y, \ t \in (-\infty, \infty)_{\mathbb{T}},$$

where $p(t) \equiv 1$ and $q(t) = e^{-t^2}$. We need to show that the assumptions in Theorem 15. It is clear that $p(t) > 0$, where $t \in (-\infty, \infty)_{\mathbb{T}}$. Furthermore we have

$$\lim_{t \to \pm\infty} |q(t)| = \lim_{t \to \pm\infty} \left| e^{-t^2} \right| = 0,$$

i.e., the assumptions of Theorem 15. Then the interval $(-\infty, 0)$ contains no point of the continuous spectrum of any, self-adjoint extension L_s of the singular operator $L_{\min}$; on the contrary, any L_s can only have at most point-eigenvalues on this interval and these can have a point of accumulation only at $\lambda = 0$.

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