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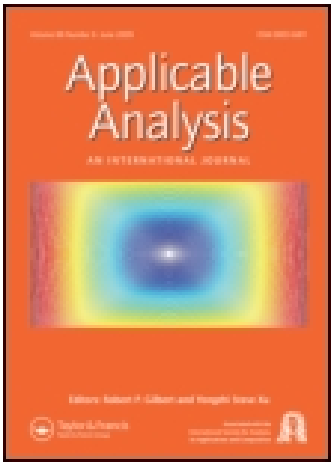

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In this study, we investigate the statistical continuity in a probabilistic normed space. In this context, the statistical continuity properties of the probabilistic norm, the vector addition and the scalar multiplication are examined.

Keywords: probabilistic normed space; strong topology; strong statistical convergence; statistical continuity

Mathematics Subject Classifications 2000: 54E70; 46S50

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1. Introduction

A probabilistic normed space (briefly, a *PN* space) is a natural generalization of an ordinary normed linear space. In a *PN* space, the norms of the vectors are represented by probability distribution functions rather than crisp numerical values. If p is an element of a *PN* space, then its norm is denoted by N_p, and the value $N_p(x)$ is interpreted as the probability that the norm of p is smaller than x, where $x \in [0, \infty]$.

PN spaces were first introduced by Šerstnev in [17] by means of a definition that was closely modelled on the theory of normed spaces. In 1993, Alsina et al. [1] presented a new definition of a *PN* space which includes the definition of Šerstnev [17] as a special case. This new definition has naturally led to the definition of the principal class of *PN* spaces, the Menger spaces, and is compatible with various possible definitions of a probabilistic inner product space. It is based on the probabilistic generalization of a characterization of ordinary normed spaces by means of a betweenness relation and relies on the tools of the theory of probabilistic metric (*PM*) spaces (see [13,14]). This new definition quickly became the standard one and it has been adopted by many authors (for instance, [3, 8–12]), who have investigated several properties of *PN* spaces. A detailed history and the development of the subject up to 2006 can be found in [15].

Our work has been inspired by Alsina et al. [2], in which the continuity properties of the probabilistic norm and the vector space operations (vector addition and scalar multiplication) are studied in detail and it is shown that a *PN* space endowed with the

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strong topology turns out to be a topological vector space under certain conditions. The aim of this article is to investigate a more general and important type of continuity, namely, the statistical continuity of the probabilistic norm and the vector space operations via the concept of strong statistical convergence, that we have recently introduced in [16]. The strong statistical convergence is a natural extension of the statistical convergence of real sequences (see [6] and [18]) to sequences in a *PM* space endowed with the strong topology. Since the study of continuity in *PN* spaces is fundamental to probabilistic functional analysis, we feel that the concept of statistical continuity in a *PN* space would provide a more general framework for the subject.

The article is organized as follows. In the second section, some preliminary concepts related to *PN* spaces and statistical convergence are presented. In the third section, the statistical continuity properties of the probabilistic norm and the vector space operations are investigated. In this context, we obtain some main results that are just parallel to the ones given in [2].

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2. Preliminaries

First we recall some of the basic concepts related to the theory of *PN* spaces. For more details we refer to [2,13,14].

Definition 2.1 A *distribution function* is a nondecreasing function F defined on $R=[-\infty,+\infty]$, with $F(-\infty)=0$ and $F(\infty)=1$.

The set of all distribution functions that are left-continuous on $(-\infty,\infty)$ is denoted by Δ.

The elements of Δ are partially ordered via

$$F \leq G \quad \text{iff } F(x) \leq G(x) \quad \text{for all } x \in R.$$

Definition 2.2 For any a in R, ε_a, the *unit step at a*, is the function in Δ given by

$$\varepsilon_a(x) = \begin{cases} 0, & -\infty \leq x \leq a \\ 1, & a < x \leq \infty \end{cases} \quad \text{for } -\infty \leq a < \infty,$$

$$\varepsilon_\infty(x) = \begin{cases} 0, & -\infty \leq x < \infty \\ 1, & x = \infty \end{cases}.$$

Definition 2.3 The distance $d_{\mathrm{L}}(F,G)$ between two functions $F,G \in \Delta$ is defined as the infimum of all numbers $h \in (0,1]$ such that the inequalities

$$F(x-h)-h \leq G(x) \leq F(x+h)+h$$

and

$$G(x-h)-h \leq F(x) \leq G(x+h)+h$$

hold for every $x \in (-(1/h),(1/h))$.

d_{L} is called the modified Lévy metric on Δ.

Definition 2.4 A *distance distribution function* is a nondecreasing function F defined on $R^+=[0,\infty]$ that satisfies $F(0)=0$ and $F(\infty)=1$, and is left-continuous on $(0,\infty)$.

The set of all distance distribution functions is denoted by Δ^+ and the metric space $(\Delta^+, d_{\mathrm{L}})$ is compact.

Definition 2.5 A *triangle function* is a binary operation τ on Δ^+, τ: $\Delta^+ \times \Delta^+ \to \Delta^+$, that is commutative, associative, nondecreasing in each place, and has ε_0 as identity.

Definition 2.6 A *probabilistic normed space* (briefly, a *PN* space) is a quadruple (S, η, τ, τ^*) where S is a real linear space, τ and τ^* are continuous triangle functions, and η is a mapping from S into the space of distribution functions Δ^+, such that – writing N_p for $\eta(p)$ – for all p, q in S, the following conditions hold:

(N1) $N_p = \varepsilon_0$ if and only if $p = \theta$, the null vector in S,
(N2) $N_{-p} = N_p$
(N3) $N_{p+q} \geq \tau(N_p, N_q)$
(N4) $N_p \leq \tau^*(N_{\alpha p}, N_{(1-\alpha)p})$, for all α in $[0, 1]$.

It follows from (N1), (N2) and (N3) that, if $\mathcal{F} : S \times S \to \Delta^+$ is defined via

$$\mathcal{F}(p, q) = F_{pq} = N_{p-q}, \tag{2.1}$$

then $(S, \mathcal{F}, \tau)$ is a *PM* space ([13], Chap. 8). Furthermore, since τ is continuous, the system of neighbourhoods $\{V_p(\lambda)\colon p \in S \text{ and } \lambda > 0\}$, where

$$V_p(\lambda) = \left\{q \in S\colon d_{\mathrm{L}}\left(F_{pq}, \varepsilon_0\right) < \lambda\right\} \tag{2.2}$$

determines a first countable and Hausdorff topology on S, called the *strong topology*. Thus, the strong topology can be completely specified in terms of the convergence of sequences.

In the following, we list some of the basic concepts related to the theory of statistical convergence and we refer to [6,7,18] for more details.

Definition 2.7 The *natural density* of a set K of positive integers is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \in K\colon k \leq n\}|$$

where $|\{k \in K\colon k \leq n\}|$ denotes the number of elements of K not exceeding n. Note that for a finite subset K of $\mathbb{N}$, we have $\delta(K) = 0$.

Notation We will be particularly concerned with integer sets having natural density zero. Thus, if (x_n) is a sequence such that (x_n) satisfies property P for all n except a set of natural density zero, then we say that (x_n) satisfies property P for '*almost all n*' and we abbreviate this by '*a.a.n*'.

Definition 2.8 A real number sequence (x_n) is said to be *statistically convergent to* $a \in \mathbb{R}$ provided that for each $\varepsilon > 0$, the set

$$K(\varepsilon) = \{n \in \mathbb{N}\colon |x_n - a| \geq \varepsilon\}$$

has natural density zero. In this case we write $stat - \lim x_n = a$.

Statistical convergence is also defined in an ordinary metric space as follows.

Definition 2.9 Let (X, ρ) be a metric space. A sequence (x_n) of points of X is said to be *statistically convergent* to an element $x \in X$, provided that for each $\varepsilon > 0$,

$$\delta\left(\left\{n \in \mathbb{N}\colon \rho(x_n, x) \geq \varepsilon\right\}\right) = 0.$$

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We denote this by $stat-\lim x_n = x$. Note that (x_n) is statistically convergent to $x \in X$ iff $stat-\lim \rho(x_n, x) = 0$; i.e. for each $\varepsilon > 0$, we have $\rho(x_n, x) < \varepsilon$ for *a.a.n.*

Using these concepts, we extend the statistical convergence to the setting of sequences in a *PN* space endowed with the strong topology as follows.

Definition 2.10 ([16]) Let (S, η, τ, τ^*) be a *PN* space. A sequence (p_n) in S is *strongly statistically convergent to a point p in S*, and we write $p_n \overset{s-stat}{\longrightarrow} p$, provided that

$$\delta\big(\{n \in \mathbb{N}: p_n \notin V_p(t)\}\big) = 0$$

for each $t > 0$. We call p as the *strong statistical limit of* (p_n).

Using (2.1) and (2.2), we can say that the following statements are equivalent:

(i) $p_n \overset{s-stat}{\longrightarrow} p$,
(ii) $\delta(\{n \in \mathbb{N}: d_{\mathrm{L}}(N_{p_n - p}, \varepsilon_0) \geq t\}) = 0$, for each $t > 0$,
(iii) $stat-\lim d_{\mathrm{L}}(N_{p_n - p}, \varepsilon_0) = 0$.

Finally, we recall the concept of statistical continuity which is an important type of sequential continuity. For a detailed discussion of statistical continuity, we refer to [4] and [5].

Definition 2.11 A function $f: \mathbb{R} \to \mathbb{R}$ is said to be *statistically continuous* at a point $x_0 \in \mathbb{R}$, if $stat-\lim x_n = x_0$ implies that $stat-\lim f(x_n) = f(x_0)$. If f is statistically continuous at each point of a set $M \subset \mathbb{R}$, then f is said to be statistically continuous on M.

In the following section, we extend the concept of statistical continuity to maps on *PN* spaces.

3. Main results

In this section we investigate the statistical continuity properties of a probabilistic norm, vector addition operation and scalar multiplication via the notion of strong statistical convergence, and present some main results.

THEOREM 3.1 *Let (S, η, τ, τ^*) be a PN space. Let S be endowed with the strong topology, and Δ^+ be endowed with the d_{L} – metric topology. Then η is a statistically continuous mapping from S into Δ^+.*

Proof It is known that the probabilistic norm η is a uniformly continuous mapping from S into Δ^+ (see [2]). Namely, for any $t > 0$ there is a $\lambda > 0$ such that, $d_{\mathrm{L}}(N_p, N_{p'}) < t$ whenever $p' \in V_p(\lambda)$. Now let (p_n) be a sequence in S such that $p_n \overset{s-stat}{\longrightarrow} p$. Then we have

$$\big\{n \in \mathbb{N}: d_{\mathrm{L}}\big(N_{p_n}, N_p\big) \geq t\big\} \subset \big\{n \in \mathbb{N}: p_n \notin V_p(\lambda)\big\}$$

for each $t > 0$. Thus, we can write

$$\delta\big(\big\{n \in \mathbb{N}: d_{\mathrm{L}}\big(N_{p_n}, N_p\big) \geq t\big\}\big) \leq \delta\big(\big\{n \in \mathbb{N}: p_n \notin V_p(\lambda)\big\}\big). \tag{3.1}$$

Since $p_n \overset{s-stat}{\longrightarrow} p$, the set on the right hand side of (3.1) has natural density zero. Hence we get

$$\delta\big(\big\{n \in \mathbb{N}: d_{\mathrm{L}}\big(N_{p_n}, N_p\big) \geq t\big\}\big) = 0$$

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for each $t>0$. Hence by Definition 2.9, we have $stat-\lim N_{p_n}=N_p$. This means that η is statistically continuous at p. Since p is arbitrary, we have that η is statistically continuous on S. ■

THEOREM 3.2 *Suppose that the hypotheses of Theorem* 3.1 *are satisfied and that* $S\times S$ *is endowed with the corresponding product topology. Then vector addition is a statistically continuous mapping from* $S\times S$ *onto* S.

Proof Let (p_n) and (q_n) be two sequences in S such that $p_n \overset{s-stat}{\longrightarrow} p$ and $q_n \overset{s-stat}{\longrightarrow} q$. Then by (N3), we can write

$$N_{(p_n+q_n)-(p+q)} \geq \tau\big(N_{p_n-p}, N_{q_n-q}\big)$$

and hence

$$d_{\mathrm{L}}\big(N_{(p_n+q_n)-(p+q)}, \varepsilon_0\big) \leq d_{\mathrm{L}}\big(\tau\big(N_{p_n-p}, N_{q_n-q}\big), \varepsilon_0\big) \tag{3.2}$$

for every $n\in\mathbb{N}$. Since the continuity of τ implies its uniform continuity, we can say that for any $t>0$ there is a $\lambda>0$ such that, $d_{\mathrm{L}}(\tau(F,G),\varepsilon_0)<t$ whenever $d_{\mathrm{L}}(F,\varepsilon_0)<\lambda$ and $d_{\mathrm{L}}(G,\varepsilon_0)<\lambda$, where $F,G\in\Delta^+$. Now let $t>0$. Then we can find a $\lambda>0$ such that, $d_{\mathrm{L}}(\tau(N_{p_n-p}, N_{q_n-q}),\varepsilon_0)<t$ (and hence $d_{\mathrm{L}}(N_{(p_n+qn)-(p+q)},\varepsilon_0)<t$ by (3.2)) whenever $p_n\in V_p(\lambda)$ (i.e. $d_{\mathrm{L}}(N_{p_n-p},\varepsilon_0)<\lambda$) and $q_n\in V_q(\lambda)$ (i.e. $d_{\mathrm{L}}(N_{q_n-q},\varepsilon_0)<\lambda$). Thus, we have

$$\begin{aligned}&\big\{n\in\mathbb{N}\colon d_{\mathrm{L}}\big(N_{(p_n+q_n)-(p+q)},\varepsilon_0\big)\geq t\big\}\\ \subset\ &\big(\big\{n\in\mathbb{N}\colon p_n\notin V_p(\lambda)\big\}\cup\big\{n\in\mathbb{N}\colon q_n\notin V_q(\lambda)\big\}\big)\end{aligned} \tag{3.3}$$

for each $t>0$. The inclusion relation (3.3) implies that

$$\delta\big(\big\{n\in\mathbb{N}\colon d_{\mathrm{L}}\big(N_{(p_n+q_n)-(p+q)},\varepsilon_0\big)\geq t\big\}\big)\leq\delta\big(\big\{n\in\mathbb{N}\colon p_n\notin V_p(\lambda)\big\}\cup\big\{n\in\mathbb{N}\colon q_n\notin V_q(\lambda)\big\}\big). \tag{3.4}$$

Since $p_n \overset{s-stat}{\longrightarrow} p$ and $q_n \overset{s-stat}{\longrightarrow} q$, each set on the right hand side of (3.4) has natural density zero, hence their union has also natural density zero. Thus, we get

$$\delta\big(\big\{n\in\mathbb{N}\colon d_{\mathrm{L}}\big(N_{(p_n+q_n)-(p+q)},\varepsilon_0\big)\geq t\big\}\big)=0$$

for each $t>0$. This shows that $(p_n+q_n)\overset{s-stat}{\longrightarrow}(p+q)$, which completes the proof. ■

COROLLARY 3.1 *The mapping* ν *from* $S\times S$ *into* Δ^+ *given by* $\nu(p,q)=N_{p+q}$ *for any* p,q *in* S *is statistically continuous.*

Proof Let us write $\nu=\eta\circ+$, where $+$ is the vector addition operation. Hence the result easily follows from Theorems 3.1 and 3.2. ■

We now investigate the statistical continuity properties of scalar multiplication, i.e. the statistical continuity properties of the mapping from $\mathbb{R}\times S$ into S given by $\mathcal{M}(\alpha,p)=\alpha p$ for any $\alpha\in\mathbb{R}$ and any $p\in S$. First of all, we will need the following lemma.

LEMMA 3.1 ([2]) *For any* α *in* $\mathbb{R}$, *any* r *in* S, *and any* $h>0$, *there is a* $\lambda>0$ *such that*, $d_{\mathrm{L}}(N_{\alpha r},\varepsilon_0)<h$ *whenever* $d_{\mathrm{L}}(N_r,\varepsilon_0)<\lambda$.

THEOREM 3.3 *The mapping* $\mathcal{M}$ *is statistically continuous in its second place, i.e. for a fixed* α *in* $\mathbb{R}$, *scalar multiplication is a statistically continuous mapping from* S *into* S.

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Proof Let $\alpha \in \mathbb{R}$ be fixed and (p_n) be a sequence in S such that, $p_n \overset{s-stat}{\longrightarrow} p$. Then by Lemma 3.1 we get

$$\{n \in \mathbb{N}: d_L(N_{p_n - p}, \varepsilon_0) < \lambda\} \subset \{n \in \mathbb{N}: d_L(N_{\alpha(p_n - p)}, \varepsilon_0) < h\}$$

for any $h>0$. Since $p_n \overset{s-stat}{\longrightarrow} p$, we have $d_L(N_{p_n - p}, \varepsilon_0) < \lambda$ for *a.a.n*. Thus, we have for each $h>0$, $d_L(N_{\alpha(p_n - p)}, \varepsilon_0) < h$ for *a.a.n*. This shows that $\alpha p_n \overset{s-stat}{\longrightarrow} \alpha p$, and hence the result. ■

However, as the following example shows, the mapping $\mathcal{M}$ need not be statistically continuous in its first place for $p \neq \theta$.

Example 3.1 (see [2]) Let S be the real line $\mathbb{R}$, viewed as a one-dimensional linear space, let $\tau = \tau_W$ and $\tau^* = \tau_M$, where τ_W and τ_M are the continuous triangle functions defined by

$$(\tau_W(F, G))(x) = \sup\{\max\{F(u) + G(v) - 1, 0\}: u + v = x\},$$
$$(\tau_M(F, G))(x) = \sup\{\min\{F(u), G(v)\}: u + v = x\}.$$

For $p \in \mathbb{R}$, define η by setting $\eta(0) = \varepsilon_0$, and

$$\eta(p) = \frac{1}{|p| + 2}\ \varepsilon_0 + \frac{|p| + 1}{|p| + 2}\ \varepsilon_\infty \quad \text{for} \quad p \neq 0.$$

It is easy to see that $(\mathbb{R}, \eta, \tau_W, \tau_M)$ is a *PN* space. Now consider the real sequence (α_n) defined by

$$\alpha_n = \begin{cases} 1 & \text{if } n = k^2 \\ \dfrac{1}{n} & \text{if } n \neq k^2 \end{cases}$$

where $k \in \mathbb{N}$. Observe that $stat - \lim \alpha_n = 0$ but $stat - \lim d_L(N_{\alpha_n p}, \varepsilon_0) \neq 0$, which shows that the mapping from $\mathbb{R}$ into S defined by $\alpha \mapsto \alpha p$ is not statistically continuous. This proves our assertion. ■

However, we see via the following lemma that the mapping $\mathcal{M}$ is statistically continuous in its first place whenever the triangle function τ^* is Archimedean, namely, τ^* admits no idempotents other than ε_0 and ε_∞.

LEMMA 3.2 ([2]) *If τ^* is Archimedean, then for any p in S such that $N_p \neq \varepsilon_\infty$ and any $h>0$, there is a $\beta>0$ such that $d_L(N_{\alpha p}, \varepsilon_0) < h$ whenever $|\alpha| < \beta$.*

THEOREM 3.4 *If (S, η, τ, τ^*) is a PN space such that τ^* is Archimedean, and if $N_p \neq \varepsilon_\infty$ for all $p \in S$, then for any fixed $p \in S$, the mapping $\mathcal{M}$ is statistically continuous in its first place.*

Proof Let $p \in S$ be fixed and (α_n) be a real sequence such that $stat - \lim \alpha_n = \alpha$. Let $h>0$ be given. Then by Lemma 3.2, we can find a $\beta>0$ such that $d_L(N_{(\gamma - \alpha)p}, \varepsilon_0) < h$ whenever $|\gamma - \alpha| < \beta$. Thus, in particular, for any $h>0$ there is a $\beta>0$ such that $|\alpha_n - \alpha| < \beta$ implies that $d_L(N_{(\alpha_n - \alpha)p}, \varepsilon_0) < h$. Hence we get

$$\{n \in \mathbb{N}: d_L(N_{(\alpha_n - \alpha)p}, \varepsilon_0) \geq h\} \subset \{n \in \mathbb{N}: |\alpha_n - \alpha| \geq \beta\}$$

for any $h>0$. Since $stat - \lim \alpha_n = \alpha$, we get

$$\delta(\{n \in \mathbb{N}: d_L(N_{(\alpha_n - \alpha)p}, \varepsilon_0) \geq h\}) = 0$$

for each $h>0$, i.e. $\alpha_n p \overset{s-stat}{\longrightarrow} \alpha p$, as desired. ■

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The following lemmas will be used in the proof of Theorem 3.5.

LEMMA 3.3 ([2]) *If $0 \le \alpha < \beta$, then $N_{\beta p} \le N_{\alpha p}$ for any p in S.*

LEMMA 3.4 ([13]) *Let τ be a continuous triangle function and $\mathcal{S}$ the set of all triples (F, G, H) in $(\Delta^+)^3$ such that*

$$F \ge \tau(H, G) \quad \text{and} \quad G \ge \tau(H, F).$$

Then for every $h > 0$ there is a $\lambda > 0$ such that if (F, G, H) is in $\mathcal{S}$ and $d_L(H, \varepsilon_0) < \lambda$, then $d_L(F, G) < h$.

THEOREM 3.5 *Suppose the hypotheses of Theorem 3.4 are satisfied. Then scalar multiplication is a jointly statistically continuous mapping from $\mathbb{R} \times S$, endowed with the natural product topology, onto S. Furthermore, the mapping $\mathcal{M}'$ from $\mathbb{R} \times S$ into Δ^+ given by $\mathcal{M}'(\alpha, p) = \eta(\alpha p)$ for any $\alpha \in \mathbb{R}$ and any $p \in S$, is also jointly statistically continuous.*

Proof Let (p_n) be a sequence in S such that $p_n \xrightarrow{s-stat} p$ and (α_n) be a real sequence such that $stat - \lim \alpha_n = \alpha$. First, let us consider the set

$$M_1 = \{n \in \mathbb{N}: |\alpha_n - \alpha| < 1\}$$

where $\delta(M_1) = 1$, by assumption. Note that we have $|\alpha_n| < |\alpha| + 1$ if $n \in M_1$. Now by (N2) and (N3) we can write

$$N_{\alpha_n p_n - \alpha p} \ge \tau\big(N_{\alpha_n(p_n - p)}, N_{(\alpha_n - \alpha)p}\big) = \tau\big(N_{|\alpha_n|(p_n - p)}, N_{(\alpha_n - \alpha)p}\big).$$

It follows by Lemma 3.3 that

$$N_{\alpha_n p_n - \alpha p} \ge \tau\big(N_{(|\alpha|+1)(p_n - p)}, N_{(\alpha_n - \alpha)p}\big)$$

if $n \in M_1$, i.e.

$$d_L\big(N_{\alpha_n p_n - \alpha p}, \varepsilon_0\big) \le d_L\big(\tau\big(N_{(|\alpha|+1)(p_n - p)}, N_{(\alpha_n - \alpha)p}\big), \varepsilon_0\big) \tag{3.5}$$

whenever $n \in M_1$. Now let $t > 0$. Since τ is uniformly continuous, we can find a $\lambda > 0$ such that

$$d_L\big(\tau\big(N_{(|\alpha|+1)(p_n - p)}, N_{(\alpha_n - \alpha)p}\big), \varepsilon_0\big) < t \tag{3.6}$$

whenever

$$d_L\big(N_{(|\alpha|+1)(p_n - p)}, \varepsilon_0\big) < \lambda \quad \text{and} \quad d_L\big(N_{(\alpha_n - \alpha)p}, \varepsilon_0\big) < \lambda.$$

Now for such a $\lambda > 0$, set

$$M_2 = \big\{n \in \mathbb{N}: d_L\big(N_{(|\alpha|+1)(p_n - p)}, \varepsilon_0\big) < \lambda\big\}$$

and

$$M_3 = \big\{n \in \mathbb{N}: d_L\big(N_{(\alpha_n - \alpha)p}, \varepsilon_0\big) < \lambda\big\}.$$

By assumption, we have $\delta(M_2) = \delta(M_3) = 1$ and thus $\delta(M_1 \cap M_2 \cap M_3) = 1$. Now for each $n \in M_1 \cap M_2 \cap M_3$ we have $d_L(N_{\alpha_n p_n - \alpha p}, \varepsilon_0) < t$ from (3.5) and (3.6). Hence

$$\delta\big(\big\{n \in \mathbb{N}: d_L\big(N_{\alpha_n p_n - \alpha p}, \varepsilon_0\big) \ge t\big\}\big) = 0,$$

which shows that $\alpha_n p_n \xrightarrow{s-stat} \alpha p$ since $t > 0$ is arbitrary. Hence the first conclusion follows.

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Now let us show that the mapping $\mathcal{M}'$ is jointly statistically continuous. Assume that $p_n \overset{s-stat}{\longrightarrow} p$ and $stat-\lim \alpha_n = \alpha$. Then we have $\alpha_n p_n \overset{s-stat}{\longrightarrow} \alpha p$, i.e. $stat-\lim d_L(N_{\alpha_n p_n - \alpha p}, \varepsilon_0) = 0$. Now by (N3), we can write

$$N_{\alpha_n p_n} \geq \tau\left(N_{\alpha_n p_n - \alpha p}, N_{\alpha p}\right)$$

and

$$N_{\alpha p} \geq \tau\left(N_{\alpha p - \alpha_n p_n}, N_{\alpha_n p_n}\right)$$

for every $n \in \mathbb{N}$. Thus, by Lemma 3.4, we can say that for any $h>0$ there is a $\lambda>0$ such that $d_L(N_{\alpha_n p_n}, N_{\alpha p}) < h$ whenever $d_L(N_{\alpha_n p_n - \alpha p}, \varepsilon_0) < \lambda$. Now using arguments similar to those of the preceding proofs, we get $stat-\lim d_L(N_{\alpha_n p_n}, N_{\alpha p}) = 0$, which shows that $\mathcal{M}'$ is jointly statistically continuous. ■

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