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Strong Statistical Convergence in Probabilistic Metric Spaces

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Abstract: In this article, we introduce the concepts of strongly statistically convergent sequence and strong statistically Cauchy sequence in a probabilistic metric (PM) space endowed with the strong topology, and establish some basic facts. Next, we define the strong statistical limit points and the strong statistical cluster points of a sequence in this space and investigate the relations between these concepts.

Keywords: Probabilistic metric space; Strong statistical cluster point; Strong statistical convergence; Strong statistical limit point; Strong statistically Cauchy sequence; Strong topology.

Mathematics Subject Classification: 54E70.

1. INTRODUCTION

The theory of probabilistic metric (PM) spaces started with Menger [11] under the name of “statistical metric spaces,” as a generalization

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of ordinary metric spaces. In this theory, the notion of distance is considered as statistical rather than deterministic. Hence instead of associating a number to a pair of points (p, q) , a distribution function F_{pq} is associated; and for any positive number x , the value $F_{pq}(x)$ is interpreted as the probability that the distance from p to q is less than x .

The theory was brought to its present state by Schweizer and Sklar [16–19], Šerstnev [22], Tardiff [24], and Thorp [25] in a series of articles. There are also many others studying on analysis of probabilistic metric spaces (see, for instance [3, 21]). A clear and detailed history of the subject up to 1983 can be found in the famous book by Schweizer and Sklar [20].

Probabilistic metric spaces have nice topological properties. Many different topological structures may be defined on a PM space. The one that has received the most attention to date is the strong topology and it is the principal tool of this article. The convergence with respect to this topology is called strong convergence. Since the strong topology is first countable and Hausdorff, it can be completely specified in terms of the strong convergence of sequences.

The aim of this article is to introduce a generalization of strong convergence, namely, the strong statistical convergence in a PM space endowed with the strong topology, and to obtain basic results. Since the study of convergence of a sequence in a PM space is very important to probabilistic analysis, we feel that the concept of strong statistical convergence in a PM space would provide a more general framework for the theory of PM spaces.

The concept of statistical convergence was first introduced for real sequences by Steinhaus [23] and developed by Fast [5]. Since then it was discussed by many authors in more general abstract spaces; for instance, in locally convex spaces [9], Banach spaces [2, 8, 14] and the fuzzy number space [1, 12]. Some applications of this notion can be found in [10, 13].

There are many pioneer works in the theory of statistical convergence (see, for instance, [2, 6, 7, 15]). In this article we will also consider the ones by Fridy [6, 7] in which the notions of statistically Cauchy sequence [6] and statistical limit points [7] were introduced for real sequences. We investigate these notions in the setting of sequences in a PM space endowed with the strong topology and try to establish basic facts related to these concepts after introducing the concept of strong statistical convergence.

2. PRELIMINARIES

In this section we present some preliminary definitions and results related to PM spaces and statistical convergence.

First we recall some of the basic concepts related to the theory of PM spaces. All the concepts listed below are studied in depth in the fundamental book [20] by Schweizer and Sklar.

Definition 2.1. A *distribution function* is a nondecreasing function F defined on $R = [-\infty, +\infty]$, with $F(-\infty) = 0$ and $F(\infty) = 1$.

The set of all distribution functions that are left continuous on $(-\infty, \infty)$ is denoted by Δ .

The elements of Δ are partially ordered via

$$F \leq G \text{ iff } F(x) \leq G(x) \text{ for all } x \in R.$$

Definition 2.2. For any $a \in [-\infty, \infty)$, ε_a , the *unit step at a* is a function in Δ , and is defined by

$$\varepsilon_a(x) = \begin{cases} 0, & -\infty \leq x \leq a \\ 1, & a < x \leq \infty \end{cases}.$$

Definition 2.3. A sequence (F_n) of distribution functions *converges weakly* to a distribution function F (and we write $F_n \xrightarrow{w} F$) if and only if the sequence $(F_n(x))$ converges to $F(x)$ at each continuity point x of F .

Definition 2.4. The distance $d_L(F, G)$ between two functions $F, G \in \Delta$ is defined as the infimum of all numbers $h \in (0, 1]$ such that the inequalities

$$F(x - h) - h \leq G(x) \leq F(x + h) + h$$

and

$$G(x - h) - h \leq F(x) \leq G(x + h) + h$$

hold for every $x \in (-\frac{1}{h}, \frac{1}{h})$.

It is known that d_L is a metric on Δ and, for any sequence (F_n) in Δ and $F \in \Delta$, we have

$$F_n \xrightarrow{w} F \text{ if and only if } d_L(F_n, F) \rightarrow 0.$$

In the sequel we will be interested in the subset of Δ consisting of those elements F that satisfy $F(0) = 0$.

Definition 2.5. A *distance distribution function* is a nondecreasing function F defined on $R^+ = [0, \infty]$ that satisfies $F(0) = 0$ and $F(\infty) = 1$, and is left continuous on $(0, \infty)$.

The set of all distance distribution functions is denoted by Δ^+ .

The function d_L is clearly a metric on Δ^+ . The metric space (Δ^+, d_L) is compact, and hence complete.

Theorem 2.1. Let $F \in \Delta^+$ be given. Then for any $t > 0$,

$$F(t) > 1 - t \text{ iff } d_L(F, \varepsilon_0) < t.$$

Note 2.1. Geometrically, $d_L(F, \varepsilon_0)$ is the abscissa of the point of intersection of the line $y = 1 - x$ and the graph of F (completed, if necessary, by the addition of vertical segments at discontinuities).

Definition 2.6. A *triangle function* is a binary operation τ on Δ^+ , $\tau: \Delta^+ \times \Delta^+ \rightarrow \Delta^+$, that is commutative, associative, nondecreasing in each place, and has ε_0 as identity.

Definition 2.7. A *probabilistic metric space* (briefly, a *PM space*) is a triple (S, \mathcal{F}, τ) where S is a nonempty set (whose elements are the *points* of the space), \mathcal{F} is a function from $S \times S$ into Δ^+ , τ is a triangle function, and the following conditions are satisfied for all p, q, r in S :

- (i) $\mathcal{F}(p, p) = \varepsilon_0$ (PM1)
- (ii) $\mathcal{F}(p, q) \neq \varepsilon_0$ if $p \neq q$ (PM2)
- (iii) $\mathcal{F}(p, q) = \mathcal{F}(q, p)$ (PM3)
- (iv) $\mathcal{F}(p, r) \geq \tau(\mathcal{F}(p, q), \mathcal{F}(q, r))$ (PM4).

In the sequel we shall denote the distribution function $\mathcal{F}(p, q)$ by F_{pq} and its values at x by $F_{pq}(x)$.

Definition 2.8. Let (S, \mathcal{F}, τ) be a *PM space*. For $p \in S$ and $t > 0$, the *strong t -neighborhood of p* is defined by the set

$$\mathcal{N}_p(t) = \{q \in S : F_{pq}(t) > 1 - t\}.$$

The collection $\mathfrak{N}_p = \{\mathcal{N}_p(t) : t > 0\}$ is called the *strong neighborhood system* at p , and the union $\mathfrak{N} = \bigcup_{p \in S} \mathfrak{N}_p$ is said to be the *strong neighborhood system for S* .

Note that we can write $\mathcal{N}_p(t) = \{q \in S : d_L(F_{pq}, \varepsilon_0) < t\}$ by Theorem 2.1.

If τ is continuous, then the strong neighborhood system \mathfrak{N} determines a *Hausdorff topology* for S . This topology is called the *strong topology for S* .

Definition 2.9. Let (S, \mathcal{F}, τ) be a *PM space*. Then for any $t > 0$, the subset $\mathcal{U}(t)$ of $S \times S$ given by

$$\mathcal{U}(t) = \{(p, q) : F_{pq}(t) > 1 - t\}$$

is called the *strong t -vicinity*.

Theorem 2.2. Let (S, \mathcal{F}, τ) be a PM space and τ is continuous. Then for any $t > 0$, there is an $\eta > 0$ such that $\mathcal{U}(\eta) \circ \mathcal{U}(\eta) \subseteq \mathcal{U}(t)$, where

$$\mathcal{U}(\eta) \circ \mathcal{U}(\eta) = \{(p, r) : \text{for some } q, (p, q) \text{ and } (q, r) \text{ are in } \mathcal{U}(\eta)\}.$$

Note 2.2. Under the hypotheses of Theorem 2.2 we can say that for any $t > 0$, there is an $\eta > 0$ such that $F_{pr}(t) > 1 - t$ whenever $F_{pq}(\eta) > 1 - \eta$ and $F_{qr}(\eta) > 1 - \eta$. Equivalently, for any $t > 0$, there is an $\eta > 0$ such that $d_L(F_{pr}, \varepsilon_0) < t$ whenever $d_L(F_{pq}, \varepsilon_0) < \eta$ and $d_L(F_{qr}, \varepsilon_0) < \eta$.

In a PM space (S, \mathcal{F}, τ) where τ is continuous, the strong neighborhood system \mathfrak{N} determines a Kuratowski closure operation which is called the *strong closure*; and for any subset A of S the strong closure of A is denoted by $k(A)$. For any nonempty subset A of S , $k(A)$ is defined by

$$k(A) = \{p \in S : \text{For any } t > 0, \text{ there is a } q \in A \text{ such that } F_{pq}(t) > 1 - t\}.$$

Remark 2.1. Throughout the rest of the article, when we speak about a PM space (S, \mathcal{F}, τ) , we always assume that τ is continuous and S is endowed with the *strong topology*.

Definition 2.10. Let (S, \mathcal{F}, τ) be a PM space. A sequence (p_n) in S is said to be *strongly convergent to a point* p in S , and we write $p_n \rightarrow p$ or $\lim p_n = p$, if for any $t > 0$, there is an integer N such that p_n is in $\mathcal{N}_p(t)$ whenever $n \geq N$.

It can easily be shown that

$$p_n \rightarrow p \text{ iff } d_L(F_{p_np}, \varepsilon_0) \rightarrow 0.$$

Similarly, a sequence (p_n) in S is called a *strong Cauchy sequence* if for any $t > 0$, there is an integer N such that (p_m, p_n) is in $\mathcal{U}(t)$ whenever $m, n \geq N$.

In the following, we list some of the basic concepts related to the theory of statistical convergence and we refer to [4, 6, 15] for more details.

Definition 2.11. The *natural density* of a set K of positive integers is defined by

$$\delta(K) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \in K : k \leq n\}|$$

where $|\{k \in K : k \leq n\}|$ denotes the number of elements of K not exceeding n . Note that for a finite subset K of \mathbb{N} , we have $\delta(K) = 0$.

The natural density may not exist for a set K , but the *upper density* of K always exists and is defined by

$$\bar{\delta}(K) = \limsup_{n \rightarrow \infty} \frac{1}{n} |\{k \in K : k \leq n\}|.$$

Notation. We will be particularly concerned with integer sets having natural density zero. Thus, if (x_n) is a sequence such that (x_n) satisfies property P for all n except a set of natural density zero, then we say that (x_n) satisfies property P for “almost all n ” and we abbreviate this by “a.a.n.”

Definition 2.12. A real number sequence (x_n) is said to be *statistically convergent* to $a \in \mathbb{R}$ provided that for each $\varepsilon > 0$, the set

$$K(\varepsilon) = \{n \in \mathbb{N} : |x_n - a| \geq \varepsilon\}$$

has natural density zero. In this case we write $\text{stat} - \lim x_n = a$.

Theorem 2.3 [15]. *Let (x_n) be a real sequence. Then $\text{stat} - \lim x_n = a$ iff there exists a set $K = \{n_1 < n_2 < \dots < n_k < \dots\} \subset \mathbb{N}$ such that $\delta(K) = 1$ and $\lim_{k \rightarrow \infty} x_{n_k} = a$.*

Statistical convergence is also defined in an ordinary metric space as follows.

Definition 2.13. Let (X, ρ) be a metric space. A sequence (x_n) of points of X is said to be statistically convergent to an element $x \in X$, provided that for each $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : \rho(x_n, x) \geq \varepsilon\}) = 0.$$

Note that (x_n) is statistically convergent to $x \in X$ iff $\text{stat} - \lim \rho(x_n, x) = 0$; i.e., for each $\varepsilon > 0$, we have $\rho(x_n, x) < \varepsilon$ for a.a.n.

Similarly, a statistically Cauchy sequence (which was first defined by Fridy [6] for real sequences) in an ordinary metric space is defined as follows.

Definition 2.14. Let (X, ρ) be a metric space. A sequence (x_n) in X is said to be a *statistically Cauchy sequence* provided that for every $\varepsilon > 0$, there exists a number $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\delta(\{n \in \mathbb{N} : \rho(x_n, x_N) \geq \varepsilon\}) = 0.$$

Theorem 2.4 (See [4]). *For a sequence (x_n) in a metric space (X, ρ) , the following conditions are equivalent:*

- (i) (x_n) is a statistically Cauchy sequence,
- (ii) For every $\varepsilon > 0$, there exists a set $D \subset \mathbb{N}$ with $\delta(D) = 0$ such that $\rho(x_m, x_n) < \varepsilon$ for any $m, n \notin D$.

3. STRONG STATISTICAL CONVERGENCE

In this section we introduce the concepts of strongly statistically convergent sequence and strong statistically Cauchy sequence in a PM space (S, \mathcal{F}, τ) , and present some main results.

Definition 3.1. Let (S, \mathcal{F}, τ) be a PM space. A sequence (p_n) in S is *strongly statistically convergent to a point p in S* , and we write $p_n \xrightarrow{stat} p$, provided that for each $t > 0$,

$$\delta(\{n \in \mathbb{N} : F_{p_n p}(t) \leq 1 - t\}) = 0.$$

We call p as the *strong statistical limit of (p_n)* .

The above definition may be restated as follows:

$$p_n \xrightarrow{stat} p \text{ iff for each } t > 0, \delta(\{n \in \mathbb{N} : p_n \notin \mathcal{N}_p(t)\}) = 0.$$

Using Theorem 2.1 and Definition 2.13, we can say that the following statements are equivalent:

- (i) $p_n \xrightarrow{stat} p$
- (ii) For each $t > 0$, $\delta(\{n \in \mathbb{N} : d_L(F_{p_n p}, \varepsilon_0) \geq t\}) = 0$
- (iii) $stat - \lim d_L(F_{p_n p}, \varepsilon_0) = 0.$

Since the natural density of a finite subset of \mathbb{N} is zero, every strongly convergent sequence is strongly statistically convergent but the converse is not true in general as can be seen in the following example.

Example 3.1. Let (S, d) be the Euclidean line and $G(x) = 1 - e^{-x}$ where $G \in \Delta^+$. Consider the simple space (S, d, G) which is generated by (S, d) and G . Then this space becomes a PM space (S, \mathcal{F}) under the continuous triangle function τ_M , which is in fact a Menger space, \mathcal{F} is defined on $S \times S$ by

$$\mathcal{F}(p, q)(x) = F_{pq}(x) = G(x/d(p, q)) = 1 - e^{-\frac{x}{d-p-q}}$$

for all $p, q \in S$ and $x \in R^+$. Here we make the convention that $G(x/0) = G(\infty) = 1$ for $x > 0$, and $G(0/0) = G(0) = 0$.

Now let (p_n) be a sequence in (S, \mathcal{F}, τ_M) defined by

$$p_n = \begin{cases} 1, & \text{if } n = k^2 \\ \frac{1}{n}, & \text{if } n \neq k^2 \end{cases}$$

where $k \in \mathbb{N}$. Now consider the function $F_{p_n 0}$ defined by

$$F_{p_n 0}(x) = \begin{cases} 1 - e^{-x}, & \text{if } n = k^2 \\ 1 - e^{-xn}, & \text{if } n \neq k^2 \end{cases}.$$

Thus, in view of Note 2.1 and Theorem 2.3 we get $\text{stat} - \lim d_L(F_{p_n 0}, \varepsilon_0) = 0$, which shows that $p_n \xrightarrow{\text{stat}} 0$.

Note that the subsequence (p_{k^2}) strongly statistically converges to 1. Thus, a subsequence of a strongly statistically convergent sequence need not strongly statistically converge to the strong statistical limit of the sequence in a PM space.

Note also that the strong statistical limit is uniquely determined since (S, \mathcal{F}, τ) is Hausdorff.

Proposition 3.1. *Let (S, \mathcal{F}, τ) be a PM space. If (p_n) and (q_n) are sequences in S such that $p_n \xrightarrow{\text{stat}} p$ and $q_n \xrightarrow{\text{stat}} q$, then we have*

$$\text{stat} - \lim d_L(F_{p_n q_n}, F_{pq}) = 0.$$

Proof. It is known that \mathcal{F} is a uniformly continuous mapping from $S \times S$ into Δ^+ if τ is continuous and S is endowed with the strong topology (see [20]). Namely, for any $t > 0$ there is an $\eta(t) > 0$ such that $d_L(F_{p'q'}, F_{p'q'}) < t$, whenever $p' \in \mathcal{N}_p(\eta)$ and $q' \in \mathcal{N}_q(\eta)$. Now assume that $p_n \xrightarrow{\text{stat}} p$ and $q_n \xrightarrow{\text{stat}} q$. Then we have for any $t > 0$,

$$\{n \in \mathbb{N} : d_L(F_{p_n q_n}, F_{pq}) \geq t\} \subseteq \{n \in \mathbb{N} : p_n \notin \mathcal{N}_p(\eta)\} \cup \{n \in \mathbb{N} : q_n \notin \mathcal{N}_q(\eta)\}$$

and hence

$$\begin{aligned} & \delta(\{n \in \mathbb{N} : d_L(F_{p_n q_n}, F_{pq}) \geq t\}) \\ & \leq \delta(\{n \in \mathbb{N} : p_n \notin \mathcal{N}_p(\eta)\} \cup \{n \in \mathbb{N} : q_n \notin \mathcal{N}_q(\eta)\}). \end{aligned}$$

Since $p_n \xrightarrow{\text{stat}} p$ and $q_n \xrightarrow{\text{stat}} q$, each set on the right hand side of the above inequality has natural density zero, hence their union has also natural density zero. Thus, we get

$$\delta(\{n \in \mathbb{N} : d_L(F_{p_n q_n}, F_{pq}) \geq t\}) = 0$$

for each $t > 0$. Hence by Definition 2.13 and the discussion that follows it, we have $\text{stat} - \lim d_L(F_{p_n q_n}, F_{pq}) = 0$. \square

Proposition 3.2. *Let (S, \mathcal{F}, τ) be a PM space and (p_n) be a sequence in S . Then (p_n) strongly statistically converges to p if, and only if, there is another sequence (q_n) such that $p_n = q_n$ for a.a.n and which strongly converges to the same limit p .*

Proof. Assume that $p_n \xrightarrow{\text{stat}} p$. Then we have $\text{stat} - \lim d_L(F_{p_n p}, \varepsilon_0) = 0$. Hence by Theorem 2.3, there exists a set $A = \{n_1 < n_2 < \dots < n_k < \dots\} \subset \mathbb{N}$ such that $\delta(A) = 1$ and $\lim_{k \rightarrow \infty} d_L(F_{p_{n_k} p}, \varepsilon_0) = 0$, i.e., (p_n)

strongly converges to p along the set A . Namely, for every $t > 0$, there is an $N(t) \in \mathbb{N}$ such that $n \geq N(t)$ and $n \in A$ imply $p_n \in \mathcal{N}_p(t)$. Now define (q_n) by $q_n = p_n$ for each $n \in A$ and $q_n = p$ for $n \notin A$. This shows that the sequence (q_n) is strongly convergent to p and $p_n = q_n$ for *a.a.n*.

Now assume that $p_n = q_n$ for *a.a.n* and $q_n \rightarrow p$. Let $t > 0$. Then for each m , we can write

$$\{n \leq m : p_n \notin \mathcal{N}_p(t)\} \subseteq \{n \leq m : p_n \neq q_n\} \cup \{n \leq m : q_n \notin \mathcal{N}_p(t)\}.$$

Since $q_n \rightarrow p$, the latter set on the right hand side contains a fixed number of integers, say $c = c(t)$. Therefore,

$$\lim_{m \rightarrow \infty} \frac{1}{m} |\{n \leq m : p_n \notin \mathcal{N}_p(t)\}| \leq \lim_{m \rightarrow \infty} \frac{1}{m} |\{n \leq m : p_n \neq q_n\}| + \lim_{m \rightarrow \infty} \frac{c}{m} = 0$$

since $p_n = q_n$ for *a.a.n*. Hence $\delta(\{n \in \mathbb{N} : p_n \notin \mathcal{N}_p(t)\}) = 0$ for each $t > 0$, which means that (p_n) is strongly statistically convergent to p . \square

Definition 3.2. Let (S, \mathcal{F}, τ) be a PM space. A sequence (p_n) in S is *strong statistically Cauchy* provided that for every $t > 0$ there exists a number $N = N(t) \in \mathbb{N}$ such that

$$\delta(\{n \in \mathbb{N} : F_{p_n p_N}(t) \leq 1 - t\}) = 0.$$

As an immediate consequence of Theorem 2.1 and Definition 3.2, we can say that (p_n) is strong statistically Cauchy iff for every $t > 0$ there exists a number $N = N(t) \in \mathbb{N}$ such that $d_L(F_{p_n p_N}, \varepsilon_0) < t$ for *a.a.n*.

Proposition 3.3. In a PM space (S, \mathcal{F}, τ) , every strongly statistically convergent sequence is also a strong statistically Cauchy sequence.

Proof. Let (p_n) be a sequence in S such that $p_n \xrightarrow{stat} p$. By Theorem 2.2 we can say that for any $t > 0$ there is an $\eta > 0$ such that $d_L(F_{p_n p_N}, \varepsilon_0) < t$ whenever $d_L(F_{p_n p}, \varepsilon_0) < \eta$ and $d_L(F_{pp_N}, \varepsilon_0) < \eta$. Since $p_n \xrightarrow{stat} p$, for every $\eta > 0$, we have $d_L(F_{p_n p}, \varepsilon_0) < \eta$ for *a.a.n*. Now choose $N = N(\eta)$ such that $d_L(F_{pp_N}, \varepsilon_0) < \eta$. Then for any $t > 0$, there is an $\eta(t) > 0$ and hence there is an $N(\eta) = N(t) \in \mathbb{N}$ such that $d_L(F_{p_n p_N}, \varepsilon_0) < t$ for *a.a.n*. This shows that (p_n) is strong statistically Cauchy. \square

Notation. For $t > 0$ and a sequence (p_n) of points in (S, \mathcal{F}, τ) , let us denote the set $\{n \in \mathbb{N} : F_{p_n p_N}(t) \leq 1 - t\}$ by $E_N(t)$, where $N \in \mathbb{N}$.

Proposition 3.4. Let (p_n) be a sequence in a PM space (S, \mathcal{F}, τ) . If (p_n) is strong statistically Cauchy, then for every $t > 0$ there exists a set $A_t \subset \mathbb{N}$ with $\delta(A_t) = 0$ such that $F_{p_m p_n}(t) > 1 - t$ for any $m, n \notin A_t$.

Proof. By Theorem 2.2, we can say that for any $t > 0$ there is an $\eta(t) > 0$ such that

$$F_{pr}(t) > 1 - t \quad \text{whenever } F_{pq}(\eta) > 1 - \eta \text{ and } F_{qr}(\eta) > 1 - \eta. \quad (3.1)$$

Now let $t > 0$ and choose $\eta = \eta(t) > 0$ such that (3.1) holds. Since (p_n) is strong statistically Cauchy, there exists an $N = N(\eta) \in \mathbb{N}$ such that

$$\delta(\{n \in \mathbb{N} : F_{p_n p_N}(\eta) \leq 1 - \eta\}) = 0.$$

Now put $A_\eta = E_N(\eta)$. Thus, we have $\delta(A_\eta) = 0$, and $F_{p_m p_N}(\eta) > 1 - \eta$ and $F_{p_n p_N}(\eta) > 1 - \eta$ for any $m, n \notin A_\eta$. Thus, for every $t > 0$ there exists a set $A_\eta = A_t \subset \mathbb{N}$ with $\delta(A_t) = 0$ such that $F_{p_m p_n}(t) > 1 - t$ for any $m, n \notin A_t$. \square

Corollary 3.1. *If (p_n) is a strong statistically Cauchy sequence in a PM space (S, \mathcal{F}, τ) , then for every $t > 0$ there exists a set $B_t \subset \mathbb{N}$ with $\delta(B_t) = 1$ such that $F_{p_m p_n}(t) > 1 - t$ for any $m, n \in B_t$.*

Proposition 3.5. *Let (S, \mathcal{F}, τ) be a PM space. If (p_n) and (q_n) are strong statistically Cauchy sequences in S , then $(F_{p_n q_n})$ is a statistically Cauchy sequence in (Δ^+, d_L) .*

Proof. Since (p_n) and (q_n) are strong statistically Cauchy, then by Corollary 3.1, for every $\eta > 0$ there exist $B_\eta, C_\eta \subset \mathbb{N}$ with $\delta(B_\eta) = \delta(C_\eta) = 1$ such that $F_{p_{m'} p_{n'}}(\eta) > 1 - \eta$ holds for any $m', n' \in B_\eta$, and $F_{q_{m''} q_{n''}}(\eta) > 1 - \eta$ holds for any $m'', n'' \in C_\eta$. Now consider the set $B_\eta \cap C_\eta$, say, D_η where $\delta(D_\eta) = 1$. Then we can say that for every $\eta > 0$ there exists a set $D_\eta \subset \mathbb{N}$ with $\delta(D_\eta) = 1$ such that $F_{p_m p_n}(\eta) > 1 - \eta$ and $F_{q_m q_n}(\eta) > 1 - \eta$ for any $m, n \in D_\eta$. Now let $t > 0$. Then there is an $\eta(t)$ and hence a set $D_\eta = D_t \subset \mathbb{N}$ with $\delta(D_t) = 1$ such that $d_L(F_{p_m q_m}, F_{p_n q_n}) < t$ for any $m, n \in D_t$ since \mathcal{F} is uniformly continuous. The rest follows from Theorem 2.4. \square

4. STRONG STATISTICAL LIMIT POINTS AND STRONG STATISTICAL CLUSTER POINTS

In this section we extend the concepts of thin subsequence, nonthin subsequence, statistical limit points and statistical cluster points of a real sequence introduced in [7] to the setting of sequences in a PM space.

Throughout the following S denotes the PM space (S, \mathcal{F}, τ) .

Definition 4.1. Let (p_n) be a sequence in S . We say that a point $p \in S$ is a strong limit point of (p_n) provided that there is a subsequence of (p_n) that strongly converges to p . We denote the set of all strong limit points of (p_n) by $L_s(p_n)$.

Definition 4.2 [See [7]]. Let (p_n) be a sequence in S and (p_{n_j}) be a subsequence of (p_n) . Denote $K = \{n_j : j \in \mathbb{N}\}$. If $\delta(K) = 0$, then we say that (p_{n_j}) is a thin subsequence of (p_n) . In case $\delta(K) > 0$ or $\delta(K)$ does not exist, i.e., $\bar{\delta}(K) > 0$, (p_{n_j}) is called a nonthin subsequence.

Definition 4.3. Let (p_n) be a sequence in S . Then an element $q \in S$ is a strong statistical limit point of (p_n) provided that there exists a nonthin subsequence of (p_n) that strongly converges to q . We denote the set of all strong statistical limit points of (p_n) by $\Lambda_s(p_n)$.

Definition 4.4. Let (p_n) be a sequence in S . Then an element $r \in S$ is a strong statistical cluster point of (p_n) provided that for every $t > 0$ we have $\bar{\delta}(\{n \in \mathbb{N} : F_{p_n r}(t) > 1 - t\}) > 0$. We denote the set of all strong statistical cluster points of (p_n) by $\Gamma_s(p_n)$.

Proposition 4.1. For any sequence (p_n) in S , we have

$$\Lambda_s(p_n) \subseteq \Gamma_s(p_n) \subseteq L_s(p_n).$$

Proof. Assume that $q \in \Lambda_s(p_n)$. Then there is a nonthin subsequence (p_{n_j}) of (p_n) that strongly converges to q , namely, $\bar{\delta}(\{n_j : j \in \mathbb{N}\}) = d > 0$. Since

$$\{n \in \mathbb{N} : F_{p_n q}(t) > 1 - t\} \supseteq \{n_j \in \mathbb{N} : F_{p_{n_j} q}(t) > 1 - t\}$$

for every $t > 0$, we have

$$\{n \in \mathbb{N} : F_{p_n q}(t) > 1 - t\} \supseteq \{n_j : j \in \mathbb{N}\} \setminus \{n_j \in \mathbb{N} : F_{p_{n_j} q}(t) \leq 1 - t\}.$$

Since $p_{n_j} \rightarrow q$, the set $\{n_j \in \mathbb{N} : F_{p_{n_j} q}(t) \leq 1 - t\}$ is finite for any $t > 0$. Thus, we have

$$\begin{aligned} \bar{\delta}(\{n \in \mathbb{N} : F_{p_n q}(t) > 1 - t\}) &\geq \bar{\delta}(\{n_j : j \in \mathbb{N}\}) \\ &\quad - \bar{\delta}(\{n_j \in \mathbb{N} : F_{p_{n_j} q}(t) \leq 1 - t\}) \\ &= d > 0. \end{aligned}$$

Thus, $\bar{\delta}(\{n \in \mathbb{N} : F_{p_n q}(t) > 1 - t\}) > 0$ for every $t > 0$, i.e., $q \in \Gamma_s(p_n)$.

Now let $q \in \Gamma_s(p_n)$ be given. Thus, we can write

$$\bar{\delta}(\{n \in \mathbb{N} : F_{p_n q}(t) > 1 - t\}) > 0$$

for every $t > 0$. This means that there are infinitely many terms of (p_n) in every strong t -neighborhood of q , i.e., $q \in L_s(p_n)$. Hence the proof is complete. \square

Proposition 4.2. *Let (p_n) be a sequence in S . If $p_n \xrightarrow{stat} p$, then $\Lambda_s(p_n) = \Gamma_s(p_n) = \{p\}$.*

Proof. Let $p_n \xrightarrow{stat} p$. Then by Definitions 3.1 and 4.4 we have $p \in \Gamma_s(p_n)$. Now assume that there exists at least one $r \in \Gamma_s(p_n)$ such that $r \neq p$. Thus, there are $t, t' > 0$ such that

$$\{n \in \mathbb{N} : F_{p_n p}(t) \leq 1 - t\} \supseteq \{n \in \mathbb{N} : F_{p_n r}(t') > 1 - t'\}$$

holds. Hence we get

$$\bar{\delta}(\{n \in \mathbb{N} : F_{p_n p}(t) \leq 1 - t\}) \geq \bar{\delta}(\{n \in \mathbb{N} : F_{p_n r}(t') > 1 - t'\}).$$

Since $p_n \xrightarrow{stat} p$, we have $\delta(\{n \in \mathbb{N} : F_{p_n p}(t) \leq 1 - t\}) = 0$, which implies that

$$\bar{\delta}(\{n \in \mathbb{N} : F_{p_n p}(t) \leq 1 - t\}) = 0.$$

Thus, there is a $t' > 0$ such that

$$\bar{\delta}(\{n \in \mathbb{N} : F_{p_n r}(t') > 1 - t'\}) = 0,$$

a contradiction to $r \in \Gamma_s(p_n)$. Therefore, we should have $\Gamma_s(p_n) = \{p\}$. On the other hand, since $p_n \xrightarrow{stat} p$, by Proposition 3.2 and by Definition 4.3 we get $p \in \Lambda_s(p_n)$. Now Proposition 4.1 yields $\Lambda_s(p_n) = \Gamma_s(p_n) = \{p\}$. \square

Proposition 4.3. *For any sequence (p_n) in S , the set $\Gamma_s(p_n)$ of strong statistical cluster points of (p_n) is strongly closed.*

Proof. Let $p \in k(\Gamma_s(p_n))$, where k denotes the strong closure. If $t > 0$, then $\Gamma_s(p_n)$ contains some point r in $\mathcal{N}_p(t)$. Choose t' so that $\mathcal{N}_r(t') \subseteq \mathcal{N}_p(t)$. Since $r \in \Gamma_s(p_n)$, we have

$$\bar{\delta}(\{n \in \mathbb{N} : p_n \in \mathcal{N}_r(t')\}) > 0,$$

which implies that

$$\bar{\delta}(\{n \in \mathbb{N} : p_n \in \mathcal{N}_p(t)\}) > 0.$$

Hence $p \in \Gamma_s(p_n)$; i.e., $k(\Gamma_s(p_n)) \subseteq \Gamma_s(p_n)$. \square

Proposition 4.4. *If (p_n) and (q_n) are sequences in S such that $p_n = q_n$ for a.a.n, then $\Lambda_s(p_n) = \Lambda_s(q_n)$ and $\Gamma_s(p_n) = \Gamma_s(q_n)$.*

Proof. Assume that $\delta(\{n \in \mathbb{N} : p_n \neq q_n\}) = 0$ and let $u \in \Lambda_s(p_n)$, say $(p)_K = (p_{n_j})$ is a nonthin subsequence of (p_n) that strongly converges to u , where $K = \{n_j : j \in \mathbb{N}\}$. Since

$$\delta(\{n : n \in K \text{ and } p_n \neq q_n\}) = 0$$

and $\bar{\delta}(K) > 0$, it follows that

$$\bar{\delta}(\{n : n \in K \text{ and } p_n = q_n\}) > 0.$$

Therefore the latter set yields a nonthin subsequence $(q)_{K'}$ of $(q)_K$ that strongly converges to u . Hence $u \in \Lambda_s(q_n)$, that is,

$$\Lambda_s(p_n) \subseteq \Lambda_s(q_n).$$

By symmetry we have $\Lambda_s(q_n) \subseteq \Lambda_s(p_n)$, hence $\Lambda_s(p_n) = \Lambda_s(q_n)$. Similarly, it can easily be shown that $\Gamma_s(p_n) = \Gamma_s(q_n)$. \square

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